OPEN PROBLEMS / RATIONAL POINTS 2025

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Abstract. This is the collection of open problems from the *Rational Points* workshop in 2025.

1 Michael Stoll

Question 1.1. Is there a (nice) curve X over $\mathbb{Q}$ of genus $g \geq 2$ such that

$$\#X(\mathbb{Q}) \geq 10^{100} g\,?$$

Remarks.

(1) The record for genus $g = 2$ is $\#X(\mathbb{Q}) \geq 642 = 321g$.
(2) Obvious lower bound for hyperelliptic curves: $\#X(\mathbb{Q}) \geq 2g + 2$.
(3) Less obvious lower bound for hyperelliptic curves (using a trick due to Mestre): $\#X(\mathbb{Q}) \geq 8g + 12$.
(4) Unlikely intersection heuristics may suggest a bound $\#X(\mathbb{Q}) \ll g + r$, where r is the Mordell-Weil rank; see [Sto19, Question 1]. So if $r \ll g$, then one may expect $\#X(\mathbb{Q}) \ll g$.

2 Nils Bruin

Question 2.1. Can we find a mock rational point which we can show to be transcendental?

(A *mock rational point* is a p-adic point that is a zero of all Chabauty functions, but not a rational point.)

Variant: Replace Jacobian by a torus.

Remark. If there are two or more Chabauty function, we expect the points cut out by them to be there for a reason; in particular, they should be algebraic. So this question is about the case when there is exactly one Chabauty function, in which case one expects a number of zeros that grows with p, and the mock rational points among them should not have any particular special properties, so are likely to be transcendental.

3 Maarten Derickx

Let

$$E\colon y^2 = 2(x^4 + 2x^2z^2 - z^4)$$

(an elliptic curve with rank 1 over $\mathbb{Q}$, with $O = (1 : 2 : 1)$).

Question 3.1. Are there infinitely many points $(x : y : z) \in E(\mathbb{Q})$ such that

$$2(x^4 + z^4) = pw^2 \qquad \text{with a prime } p \text{ and some } w \in \mathbb{Z}?$$

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Remark. This is a special case of a family of similar questions. In the concrete case, a solution is known with a rather large prime p.

Ignore the square; then the "probability" for the value N to be (more or less) a prime is about $1/\log N$. The logarithmic height of the coordinates of multiples of a generator grows quadratically, so the expected number of positive cases in the rank 1 case should be something like

$$\sum_{n>0} \frac{1}{cn^2} < \infty\,.$$

For rank ≥ 2, the corresponding series diverges, though.·

4 Davide Lombardo

Consider the plane quartic

$$x^4 + 3x^3y - 3x^2yz - 3x^2z^2 + 6xy^3 - 6xy^2z + 3xyz^2 - 2xz^3 + 4y^4 + 2y^3z - 5yz^3 = 0\,.$$

(The twist $X(7)_{392c1}$ of $X(7)$, a.k.a. the Klein Quartic.) Its Jacobian has a nontrivial endomorphism defined over a quadratic extension.

Determine its rational points!

This would finish the classification of 7-adic Galois representations coming from elliptic curves over $\mathbb{Q}$.

Drew Sutherland: The endomorphism field is the splitting field of

$$x^8 - 2x^7 + 7x^4 - 14x^2 + 8x + 5$$

(https://www.lmfdb.org/NumberField/8.2.10330523392.1)

The geometric Picard number is 3, but the classes are not defined over $\mathbb{Q}$, so standard Quadratic Chabauty does not apply. The Jacobian does not decompose over any field of degree ≤ 24. There are at least four rational points.

5 David Angdinata

Question 5.1. Let C be a singular projective curve over an *imperfect* field K given by a Weierstrass equation. What is the structure of the group of non-singular rational points $C_{ns}(K)$?

See here on MO.

Over perfect fields, this is known ($\mathbb{G}_m$ or a quadratic twist, or $\mathbb{G}_a$); see, e.g., Silverman.

Want a general description (for the purpose of formalizing the theory).

6 Carlo Pagano

Question 6.1. Let K be a number field. Can you find a cubic non-singular polynomial $f(X)$ in $K[X]$ and a non-constant polynomial $g(T)$ in $K[T]$ totally split such that the curve $g(T)y^2 = f(X)$ has rank precisely 2 over $K(T)$?

(This would $\approx$ imply that there are infinitely many elliptic curves of rank exactly 2 over K, in case one had such an example with also f totally split over K.)

Even open for $K = \mathbb{Q}$ (an example for this case would already be nice).

The desired result on elliptic curves of rank 2 is known for some number fields.

7 Pip Goodman

Let M be an imaginary quadratic field of class number 1.

Question 7.1. Does there exist an abelian threefold defined over $\mathbb{Q}$ whose geometric endomorphism algebra is isomorphic to M?

Remarks.

(1) If such a threefold exists, M is forced to have class number 1 (so $M = \mathbb{Q}(\sqrt{-d})$ for $d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$).
(2) Examples are known for $M = \mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 7$. (Easy for $d = 1, 3$.)
(3) The field of definition of the endomorphisms is necessarily M.

8 Jack Thorne

Question 8.1. Can one prove the Lang-Silverman Conjecture for odd hyperelliptic curves of some fixed genus (≥ 2) over a global function field K (e.g., $\mathbb{F}_q(t)$)?

Concretely, is there a constant C such that $\hat{h}(P) \geq C \deg(D)$ for all non-torsion points $P \in J(K)$, where J is the Jacobian and D is the discriminant divisor of the curve?

For elliptic curves, this follows from the ABC conjecture (Hindry-Silverman), which is known for function fields.

Michael Stoll: There are examples of (even degree) genus 2 curves over $\mathbb{Q}(t)$ with a point of rather small positive canonical height:

$$y^2 = 9(4t+1)^2x^6 - 24(4t+1)(t+2)x^4 - 48(4t+1)(t-1)x^3 + 16(t-2)^2x^2 + 64(t+1)(t-2)x + 64(t+1)^2$$

with $\hat{h}(P) = \frac{1}{840}$.

Drew Sutherland: What about places of bad reduction for the curve, but good reduction for the Jacobian?

9 Sam Frengley

Question 9.1. Another genus 3 curve: $X_{ns}^{+}(24)$. Can we determine its rational points?

References

[Sto19] Michael Stoll, *Uniform bounds for the number of rational points on hyperelliptic curves of small Mordell-Weil rank*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 3, 923–956, DOI 10.4171/JEMS/857. MR3908770 ↑4

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