Adelic Mordell-Lang and the Brauer-Manin obstruction

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Notation

Throughout the talk:

- A is an abelian variety over a global field k.
- $X \subset A$ is a closed subvariety.
- $X(\mathbb{A}_k)_\bullet = \prod_v X(k_v)_\bullet$ are the adelic points.
  (modified at archimedean places)
- $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$ is the Brauer set of X.
- $\overline{X(k)}$ is the topological closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$.

We have

$$\overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} \subset X(\mathbb{A}_k)_\bullet$$

$$\overset{?}{=} \qquad \neq$$

The Main Result

Theorem [C.-Voloch]

Let $X \subset A$ be a closed subvariety of an abelian variety over a global **function** field. Assume

- $A_{\overline{k}}$ has no nonzero isotrivial quotient, and
- $\text{Ш}(A)_{\text{div}} = 0$.

Then

$$\overline{X(k)} = X(\mathbb{A}_k)^{\text{Br}}.$$

In other words, **Brauer-Manin is the only obstruction to weak approximation for X.**

- This generalizes [Poonen-Voloch 2010] which proved this for $X_{\overline{k}}$ 'coset free' with an additional hypothesis on A
- The assumption $\text{Ш}(A)_{\text{div}} = 0$ is not needed for X coset free, but is necessary for $X = A$.

Mordell-Lang

Definition

- A **coset** in A is a subvariety of the form $C = a + A'$ where $a \in A(\overline{k})$ and $A' \subset A$ is an abelian subvariety. We insist that A' and C are defined over k, so that C is a torsor under A'.
- X is **coset free** if it does not contain any positive dimensional cosets.

Mordell-Lang Conjecture [Faltings, Hrushovski]

There is a finite union of cosets $Y \subset X$ such that

$$X(k) = Y(k)\,.$$

(assuming no nonzero isotrivial quotient in the function field case)

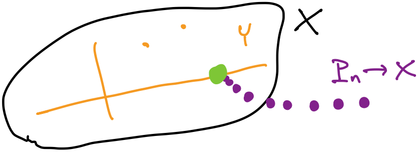
Adelic Mordell-Lang Conjecture

AML Conjecture

There is a finite union of cosets $Y \subset X$ such that

$$X(\mathbb{A}_k)_\bullet \cap \overline{A(k)} \subset Y(\mathbb{A}_k)_\bullet$$

- Mordell-Lang says there is a special subvariety Y which contains all of the the rational points of X.
- Adelic Mordell-Lang says that any sequence of rational points on A approaching X must approach Y.



- For coset free X, this was stated by Stoll over number fields and proved by Poonen-Voloch over function fields.

AML holds over function fields

Theorem [C.-Voloch]

AML holds for closed subvarieties of abelian varieties over a global function fields (with no nonzero isotrivial quotient).

- Proofs of **ML** over function fields tend to give a stronger result saying rational points v-adically close to X must be v-adically close to a special subvariety.
- Poonen-Voloch used Hrushovski's proof of **ML**, but needed the coset free hypothesis to control how the special subvariety depends on v.
- There is a different proof of Mordell-Lang by Pink/Rössler/Wisson using algebro-geometric methods. We deduce the theorem from this.

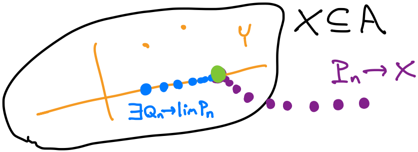
The Mordell-Weil Sieve conjecture

- The following has been asked/conjectured in various forms by Scharaschkin, Skorobogatov, Stoll, Poonen, Voloch, ...

MWS Conjecture

For a closed subvariety $X \subset A$ we have

$$X(\mathbb{A}_k)_\bullet \cap \overline{A(k)} = \overline{X(k)}\,.$$



- Assuming $Ш(A)_{\mathrm{div}} = 0$, **MWS** implies that Brauer-Manin is the only obstruction to weak approximation for X.
- Note: **MWS** $\Rightarrow$ **AML**, since $\overline{X(k)} = \overline{Y(k)}$ by **ML.**
- To get the 'main result' we prove: **AML** $\Rightarrow$ **MWS.**

AML implies MWS

Conjectures Restated:

$$\textbf{AML}: \quad X(\mathbb{A}_k)_\bullet \cap \overline{A(k)} \subset Y(\mathbb{A}_k)_\bullet \cap \overline{A(k)} = \overline{Y(k)} \subseteq \overline{X(k)}$$

$$\textbf{MWS}: \quad X(\mathbb{A}_k)_\bullet \cap \overline{A(k)} = \overline{X(k)}$$

Theorem [C.-Voloch]

If $Y \subset A$ is a finite union of cosets, then $Y(\mathbb{A}_k)_\bullet \cap \overline{A(k)} = \overline{Y(k)}$.

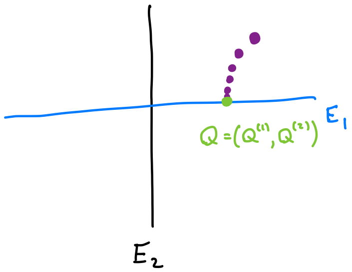
Corollary

AML implies **MWS**.

- For Y of dimension 0 this was proved over number fields by Stoll and over function fields by Poonen-Voloch. This gave **AML** implies **MWS** for $X_{\overline{k}}$ coset free.

Example: $Y = E_1 \cup E_2 \subset A = E_1 \times E_2$

- Let $P_n \in A(k)$ with $P_n \to Q \in Y(\mathbb{A}_k)$, $Q = (Q^{(1)}, Q^{(2)})$.
- We want to show $Q \in \overline{Y(k)} = \overline{E_1(k)} \cup \overline{E_2(k)}$



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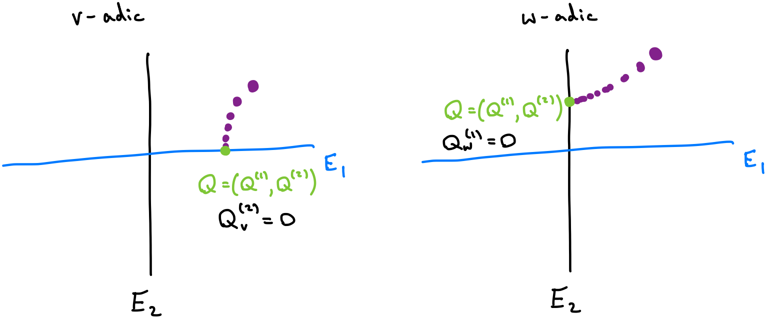

- $Q \in Y(\mathbb{A}_k)$ means $\forall v\, \exists i$ such that $Q_v^{(i)} = 0$.

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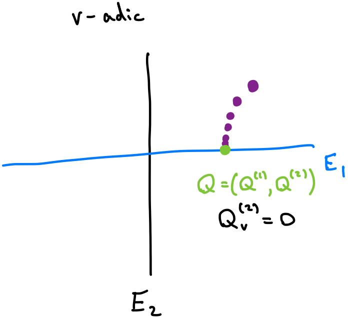

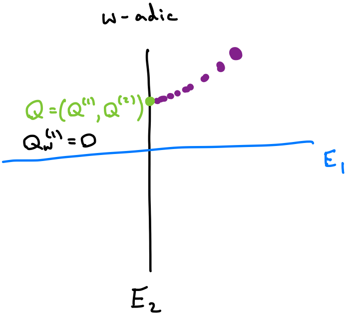

- $Q \in Y(\mathbb{A}_k)$ means $\forall v\, \exists i$ such that $Q_v^{(i)} = 0$.
- $Q \in \overline{E_1(k)} \cup \overline{E_2(k)}$ requires $\exists i\, \forall v$ we have $Q_v^{(i)} = 0$.

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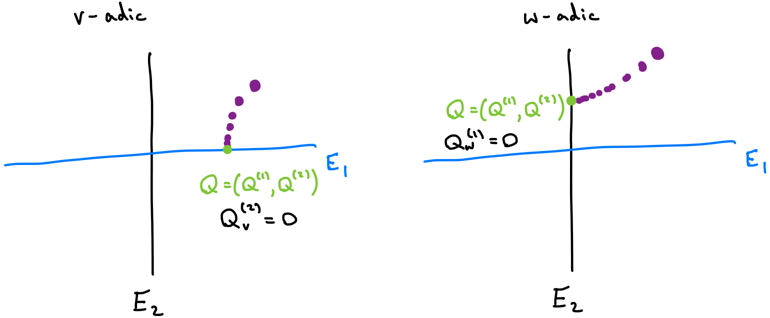

- $Q \in Y(\mathbb{A}_k)$ means $\forall v\, \exists i$ such that $Q_v^{(i)} = 0$.
- $Q \in \overline{E_1(k)} \cup \overline{E_2(k)}$ requires $\exists i\, \forall v$ we have $Q_v^{(i)} = 0$.
- How would you construct a counterexample?
  - Choose $Q^{(1)} \in \overline{E_1(k)}$ that is 0 at lots of primes, but not all.
  - Do same for $Q^{(2)}$, hoping ‘lots’ for both covers all primes.
  - Then $(Q^{(1)}, Q^{(2)}) \in Y(\mathbb{A}_k)$, but not in $\overline{E_1(k)} \cup \overline{E_2(k)}$.

Key Lemma

Lemma

Suppose $Q_1, \ldots, Q_r \in \overline{A(k)} \subset A(\mathbb{A}_k) = \prod A(k_v)$ are nonzero. Then there exists $m \geq 1$ and nonarchimedean v such that none of the Q_i has trivial image in the pro-m completion $A(k_v)^{(m)}$.

- The proofs of Stoll and Poonen-Voloch (coset free case) used this in the case $r = 1$.
- To generalize to $r > 1$ we use
  - A result of Serre about the image of Galois in $\mathrm{Aut}(T_\ell(A))$
  - Ideas of Stoll (for $\ell \neq p$)
  - Ideas of Poonen-Voloch and Rössler (for $\ell = p$)
  - Combine these (for $r > 1$ can no longer take $m = p$)
  - Chebotarev density theorem

Summary

Conjectures Restated:

$$\textbf{AML}: \quad X(\mathbb{A}_k)_\bullet \cap \overline{A(k)} \subset Y(\mathbb{A}_k)_\bullet$$
$$\textbf{MWS}: \quad X(\mathbb{A}_k)_\bullet \cap \overline{A(k)} = \overline{X(k)}$$

Theorems (restated)

- **Thm 1**: **AML** holds over function fields (nonisotrivial case).
- **Thm 2**: **AML** $\Rightarrow$ **MWS** (over all global fields)

Corollary

If $X \subset A$ is a closed subvariety over a global function field such that A has no positive dimensional isotrivial quotient, then

- **MWS** holds for $X \subset A$;
- If $\text{Ш}(A)_{\text{div}} = 0$, then Brauer-Manin is the only obstruction to weak approximation for X.

Adelic Mordell-Lang (Selmer version)

$$\overline{A(k)} \subset \widehat{\mathrm{Sel}}(A) = \varprojlim_n \mathrm{Sel}^n(A)$$

AML-Sel Conjecture

There exists a finite union of cosets $Y \subset X$ such that

$$X(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(A) \subset Y(\mathbb{A}_k)_\bullet$$

Theorem

If **AML-Sel** holds for $X \subset A$, then the following are equivalent:

1. $\overline{X(k)} \neq X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$;
2. X contains a coset $C = a + A'$ which represents a nontrivial divisible element in $Ш(A')$.

The nonisotrivial hypothesis

- Let D be a curve of genus ≥ 2 over $\mathbb{F}_q$.
- Let $k = \mathbb{F}_q(D)$.
- Let $X = D_k$.
- Then $A = \mathrm{Jac}(X) = \mathrm{Jac}(D)_k$ is isotrivial and $X \subset A$.
- X is coset free, but $X(k)$ is infinite since

$$X(k) = \mathrm{Mor}(\mathrm{Spec}(\mathbb{F}_q(D)), D_{\mathbb{F}_q(D)}) = \mathrm{Mor}(D, D) \supset \{F^i : i \geq 0\}$$

- So **ML** and **AML** (as stated above) do not hold.

Question

Does **MWS** hold for $X \subset A$?