Workshop Program 'Rational Points 2022'

(tentative)

Sunday, March 27

	Arrival
until 14:00	*Restaurant is open*
14:30–18:00	*Reception is open*
from 18:00	*Keys available at the restaurant*
18:30–21:30	*Dinner*

Monday, March 28

08:00–09:15	*Breakfast*
09:20–09:30	Michael Stoll: *Opening and Welcome*
09:30–10:30	Tom Fisher: **Computing the Cassels-Tate pairing on 2-Selmer groups of genus 2 Jacobians**
10:30–11:30	*Coffee*
11:30–12:00	Himanshu Shukla: **Computing Cassels-Tate pairing on the 2-Selmer group of odd-degree hyperelliptic curves**
12:30–13:30	*Lunch*
15:00–16:00	*Coffee*
16:00–16:45	Timo Keller: **Verification of strong BSD for abelian surfaces over $\mathbb{Q}$**
17:00–17:30	Masahiro Nakahara: **The elliptic sieve and Brauer groups**
17:45–18:15	Ludwig Fürst: **Explicit methods for hyperelliptic genus 4 Kummer varieties**
18:30–19:30	*Dinner*

Tuesday, March 29

08:00–09:15	*Breakfast*
09:30–10:30	Philipp Habegger: **Uniformity for the Number of Rational Points on a Curve (I)**
10:30–11:15	*Coffee*
11:15–12:15	Ziyang Gao: **Uniformity for the Number of Rational Points on a Curve (II)**
12:30–13:30	*Lunch*
15:00–16:00	*Coffee*
16:00–16:45	Nirvana Coppola: **Coleman integrals over number fields**
17:00–17:30	Steffen Müller: ***p*-adic Arakelov theory on abelian varieties and quadratic Chabauty**
17:45–18:15	Stevan Gajović: **Symmetric Chabauty for cubic points on certain modular curves**
18:30–19:30	*Dinner*
20:00–?	**Open Problems**

Wednesday, March 30

08:00–09:15	*Breakfast*
09:30–10:30	Isabel Vogt: **Obstructions to rationality of conic bundle threefolds**
10:30–11:15	*Coffee*
11:15–12:15	Bianca Viray: **Quadratic points on intersections of quadrics**
12:30–13:30	*Lunch*
	Free Afternoon, *Coffee* and *Dinner* on request

Thursday, March 31

08:00–09:15	*Breakfast*
09:30–10:00	Carlos Rivera: **Persistence of the Brauer-Manin obstruction on cubic surfaces**
10:15–10:45	Stephan Elsenhans: **Point Counting on K3 surfaces**
10:45–11:30	*Coffee*
11:30–12:15	Ulrich Derenthal: **Integral points on singular del Pezzo surfaces**
12:30–13:30	*Lunch*
15:00–16:00	*Coffee*
16:00–16:45	Maarten Derickx: **Explicit descent using étale morhphisms between modular curves**
17:00–17:30	Filip Najman: **Quadratic points on bielliptic modular curves**
17:45–18:15	Ari Shnidman: **Ranks of abelian varieties in isotrivial families**
18:30–19:30	*Dinner*

Friday, April 1

08:00–09:15	*Breakfast*
09:30–10:30	Abbey Bourdon: **Towards a classification of sporadic j-invariants**
10:30–11:15	*Coffee*
11:15–12:00	Michael Stoll: **Rational points on and BSD for a curve of genus 4**
12:30–13:30	*Lunch*
	Free Afternoon, *Coffee* and *Dinner* on request

Saturday, April 2

07:30–09:30	*Breakfast*
	Departure

Abstracts

Efthymios Sofos (talk postponed): **The geometric large sieve and Diophantine stability**

Recently, Mazur and Rubin studied Diophantine stability: assume we are given a pointless variety defined over the rationals $\mathbb{Q}$. Over which number field extensions $K/\mathbb{Q}$ does the variety obtain a rational point? In joint work with Carlo Pagano, we fix the extension and vary the equation. Namely, by developing a new combination of Linnik's large sieve and Ekedahl's geometric sieve we prove asymptotics for the probability that for two random integers a, b in an expanding box the Hilbert symbol $(a, b)_K$ is 1, where K is a fixed number field extension of $\mathbb{Q}$.

Tom Fisher: **Computing the Cassels-Tate pairing on 2-Selmer groups of genus 2 Jacobians**

In her thesis completed last year, my student Jiali Yan gave a practical method for computing the Cassels-Tate pairing on the 2-Selmer group of the Jacobian of a genus 2 curve all of whose Weierstrass points are rational. She also gave a second method without any assumption on the Weierstrass points, but instead assuming we can find a rational point on a certain twisted Kummer surface. The two methods can be thought of as generalising methods of Cassels and Donnelly in the elliptic curve case. I will describe some recent refinements that significantly improve the practicality of the second method.

Himanshu Shukla: **Computing Cassels-Tate pairing on the 2-Selmer group of odd-degree hyperelliptic curves**

In this talk I will talk about the computation of the Cassels-Tate pairing on the 2-Selmer group associated with the Jacobian variety of an odd-degree hyperelliptic curve defined over a number field k. The computation is based on the Albanese-Albanese definition of the pairing given by Poonen and Stoll. I will show that computing the pairing in the above case is equivalent (in terms of computational hardness) to trivializing some 2-cocycles which represent the trivial element of the Brauer group of k. I will also talk about a conditional algorithm inspired from the computation in the elliptic curve case, which seems to work for all practical purposes.

Timo Keller: **Verification of strong BSD for abelian surfaces over $\mathbb{Q}$**

The strong Birch–Swinnerton-Dyer conjecture and in particular the exact order of the Shafarevich–Tate group for abelian varieties over the rationals has only been known for elliptic curves (dimension 1) or in higher dimension where the conjecture could be reduced to dimension 1. We give the first absolutely simple examples of dimension 2 where the conjecture can be verified:

Let X be (1) a quotient of the modular curve $X_0(N)$ by a subgroup generated by Atkin-Lehner involutions such that its Jacobian J is an absolutely simple modular abelian surface, or, more generally, (2) an absolutely simple factor of $J_0(N)$ isomorphic to the Jacobian J of a genus-2 curve X. We prove that for all such J from (1), the Shafarevich–Tate group of J is trivial and satisfies the strong Birch–Swinnerton-Dyer conjecture. We further indicate how to verify strong BSD in the cases (2) in principle and in many cases in practice.

To achieve this, we compute the image and the cohomology of the mod-$\mathfrak{p}$ Galois representations of J, show effectively that almost all of them are irreducible and have maximal image, make the Heegner points Euler system of Kolyvagin–Logachev effective, compute the Heegner points and Heegner indices, compute the $\mathfrak{p}$-adic L-function, and perform $\mathfrak{p}$-descents. Because many ingredients are involved in the proof, we will give an overview of the methods involved and give more details regarding the computation of the Galois representations and the Heegner points as these are the main input for our theorem on the explicit finite support of the Shafarevich–Tate group.

This is joint work with Michael Stoll.

Masahiro Nakahara: **The elliptic sieve and Brauer groups**

A theorem of Serre states that almost all plane conics over $\mathbb{Q}$ have no rational point. We prove an analogue of this for families of conics parameterized by elliptic curves using elliptic divisibility sequences and a version of the Selberg sieve for elliptic curves. We also give more general results for specialisations of Brauer groups, which yields applications to norm form equations. This is joint work with Subham Bhakta, Daniel Loughran, and Simon Rydin Myerson.

Ludwig Fürst: **Explicit methods for hyperelliptic genus 4 Kummer varieties**

An explicit construction of the Kummer variety of a hyperelliptic curve has only been done for genus up to 3. As part of my PhD thesis I do such an explicit construction in the case of $g = 4$. In this talk I give an overview which methods have been made explicit and what functionality can be provided in the Magma language such as an specific embedding of the Kummer into $\mathbb{P}^{15}$, duplication polynomials or an algorithm to compute the canonical height of points. I will also go into problems encountered in the research. This work was done under supervision of M. Stoll and S. Müller.

Philipp Habegger and Ziyang Gao:
Uniformity for the Number of Rational Points on a Curve

Part I (Habegger): *By Faltings's Theorem, formerly known as the Mordell Conjecture, a smooth projective curve of genus at least 2 that is defined over a number field K has at most finitely many K-rational points. Votja later gave a new proof. Several authors, including Bombieri, David-Philippon, de Diego, Rémond, and Vojta, obtained upper bounds for the number of K-rational points using Vojta's approach. I will discuss joint work with Vesselin Dimitrov and Ziyang Gao where we show that the number of K-rational points is bounded from above as a function of K, the genus, and the rank of the Mordell-Weil group of the curve's Jacobian. I will also discuss some further developments by Kühne, Yuan-Zhang, and Yuan. Our work uses Vojta's approach. I will discuss the new ingredient, an inequality for the Néron-Tate height in a family of abelian varieties, and how it relates to the general strategy.*

Part II (Gao): *I will continue reporting on the joint work with Vesselin Dimitrov and Philipp Habegger. I will also talk about its generalization to high-dimensional subvarieties of abelian varieties, known as the Uniform Mordell-Lang Conjecture (joint with Tangli Ge and Lars Kühne). In the first talk, Philipp Habegger explained the history and the recent results on the number of rational points on curves of genus at least 2, especially the uniformity result of Dimitrov-Gao-Habegger. The framework of the proof was explained: apart from classical results, the key point is a height inequality. The proof of this height inequality was also sketched. It remains to show the non-degeneracy of a particular subvariety of the universal abelian variety.*
In this talk, I will focus on explaining how to answer this question of non-degeneracy. In particular, I will explain the bi-algebraic geometry associated with the universal abelian variety, the Ax-Schanuel theorem and its application to studying the non-degeneracy, both for curves and for high-dimensional subvarieties of abelian varieties.

Nirvana Coppola: **Coleman integrals over number fields**

One of the deepest mathematical results is Faltings's Theorem on the finiteness of rational points on an algebraic curve of genus $g \geq 2$*. A much more difficult question, still not completely answered, is whether given a curve of genus* $g \geq 2$*, we can find all its rational points, or, more in general, all points defined over a certain number field. An entire (currently very active!) area of research is devoted to find an answer to such questions, using the "method of Chabauty".*

In this seminar, I will talk about one of the first tools employed in Chabauty method, namely Coleman integrals, which Coleman used to compute an explicit bound on the number of rational points on a curve. After explaining how this is defined, I will give a generalisation of this definition for curves defined over number fields, and explain how to explicitly compute these integrals. This is based on an ongoing project, which started during the Arizona Winter School 2020, joint with E. Kaya, T. Keller, N. Müller, S. Muselli.

Steffen Müller: p**-adic Arakelov theory on abelian varieties and quadratic Chabauty**

TBA

Stevan Gajović: **Symmetric Chabauty for cubic points on certain modular curves**

We formulate a generalised Symmetric Chabauty criterion, building on work of Siksek, that allows us to compute the cubic points on certain modular curves $X_0(N)$ *having infinitely many quadratic points. This is joint work with Josha Box and Pip Goodman.*

Isabel Vogt: **Obstructions to rationality of conic bundle threefolds**

In this talk I'll discuss joint work with Sarah Frei, Lena Ji, Soumya Sankar and Bianca Viray on the problem of determining when a conic bundle over $\mathbb{P}^2$ *is birational (over the field of definition!) to* $\mathbb{P}^3$*. As a consequence of our work, when the ground field is the real numbers, we show that neither the topological obstruction nor the refined intermediate Jacobian obstruction is sufficient to determine rationality.*

Bianca Viray: **Quadratic points on intersections of quadrics**

TBA

Carlos Rivera: **Persistence of the Brauer-Manin obstruction on cubic surfaces**

Let X *be a smooth cubic hypersurface over a field* k*. Cassels and Swinnerton-Dyer have conjectured that* X *has a* k*-rational point as soon as it has a* 0*-cycle of degree* 1 *or, equivalently, as soon as* X *has a closed point of degree prime to* 3*. In 1974, D. Coray showed several results in this direction including, in the case of cubic surfaces, that the existence of a closed point of degree prime to* 3 *implies the existence of a closed point of degree* 1*,* 4 *or* 10*. In this talk, for* k *a global field and* X *a cubic surface, we will show that a Brauer-Manin obstruction to the existence of* k*-points on* X *will persist over every extension* L/k *with degree prime to* 3*. Therefore proving that the conjecture of Colliot-Thélène and Sansuc on the sufficiency of the Brauer-Manin obstruction for cubic surfaces implies the conjecture of Cassels and Swinnerton-Dyer in this case. This is joint work with Bianca Viray.*

Stephan Elsenhans: **Point Counting on K3 surfaces**

Counting points on varieties over finite fields is a classical topic of arithmetic geometry. In this talk I will report on recent progress in point counting on K3-surfaces over finite fields. The method can be used for a numerical confirmation of the Sato-Tate conjecture.

Ulrich Derenthal: **Integral points on singular del Pezzo surfaces**

The classical problem of finding integral solutions to Diophantine equations can be phrased, in geometric language, as the question of integral points on the corresponding algebraic varieties. For example, pairs of quadratic forms in five variables define (possibly singular) quartic del Pezzo surfaces, which often contain infinitely many integral points (with respect to a boundary such as a line or a singularity). For a certain singular quartic del Pezzo surface, we prove asymptotic formulas for the number of integral points of bounded height (with respect to all reasonable boundaries), using the universal torsor method combined with analytic techniques. This can be interpreted as an instance of an integral version of Manin's conjecture, from the point of view of Chambert-Loir and Tschinkel. (Joint work with Florian Wilsch.)

Maarten Derickx: **Explicit descent using étale morhphisms between modular curves**

Both Mazur and Momose have obtained important results regarding rational points on the classical modular curves $X_0(p)$*. One of the key parts in both their works is a careful study of properties of the isogeny character that one can associate to a cyclic isogeny between elliptic curves. So it seems likely that this approach to rational points on modular curves is really particular to modular curves. Indeed nowhere in the work of Mazur and Momose or the subsequent literature this study of isogeny characters is linked to well-known rational points techniques. However, it turns out that this careful study of isogeny characters can be interpreted as an explicit version of descent. In this talk I will explain how one can translate the study of rational points on* $X_0(p)$ *using descent into a study of isogeny characters. Additionally a new application will be given, namely the uniform boundedness of unramified isogeny characters. This uniform boundedness is a generalisation of Merel's uniform boundedness of torsion points, and a small first step in the direction of uniform boundedness of isogenies.*

Filip Najman: **Quadratic points on bielliptic modular curves**

In previous work P. Bruin and I, Ozman and Siksek, and Box described all the quadratic points on the modular curves of genus $2 \leq g(X_0(n)) \leq 5$*. Since all the hyperelliptic curves* $X_0(n)$ *are of genus* ≤ 5 *and as a curve can have infinitely many quadratic points only if it is either of genus* ≤ 1*, hyperelliptic or bielliptic, the question of describing the quadratic points on the bielliptic modular curves* $X_0(n)$ *naturally arises; this question has recently also been posed by Mazur.*

We answer Mazur's question completely and describe the quadratic points on all the bielliptic modular curves $X_0(n)$*, for which this has not been done already. The values of* n *that we deal with are* $n = 60, 62, 69, 83, 89, 92, 94, 95, 101, 119$ *and* 131*; the curves* $X_0(n)$ *are of genus up to* 11*. The two main methods we use are the relative symmetric Chabauty method of Box and Siksek, and an application of a moduli description of* $\mathbb{Q}$*-curves of degree* d *with an independent isogeny of degree* m*, which reduces the problem to finding the rational points on several quotients of modular curves. This is joint work with my PhD student Borna Vukorepa.*

Ari Shnidman: **Ranks of abelian varieties in isotrivial families**

I'll discuss recent work (with Weiss and Alpöge-Bhargava) on ranks of abelian varieties in various kinds of twist families. We apply geometry-of-numbers methods (and sometimes the circle method) to count integral orbits in certain representations (G, V)*, in order to compute the average size of various Selmer groups. As applications, we deduce average rank bounds and prove the existence (in certain cases) of many twists with rank 0. The most novel aspect of these works is that we can use one representation to study very different families of abelian varieties. For example, we show that the average rank in any cubic twist family of elliptic curves* $E_m\colon y^2 = x^3 + km^2$ *is at most* 1.5*. We also show that the average rank in the cubic twist family of simple abelian surfaces* $A_m = \mathrm{Prym}(my^3 = x^4 + ax^2 + b)$ *is at most 3. Both proofs use* $G = \mathrm{SL}_2^2$ *and* $V = \mathbb{Z}^2 \otimes \mathrm{Sym}^3 \mathbb{Z}^2$*. We will explain how to axiomatize these types of results (to the extent we know how).*

Abbey Bourdon: **Towards a classification of sporadic** j**-invariants**

We say a point x *on a curve* C *is sporadic if there exist only finitely many points on* C *of degree at most* $\deg(x)$*. In the case where* C *is the modular curve* $X_1(N)$*, a non-cuspidal sporadic point can be thought of as corresponding to an elliptic curve with a point of order* N *defined over a number field of unusually low degree. Since every elliptic curve with complex multiplication gives rise to a sporadic point on* $X_1(N)$ *for some positive integer* N*, we will primarily be interested in characterizing those non-CM elliptic curves which produce sporadic points. In this case, the problem is tied to several open questions in the arithmetic of elliptic curves, including Serre's Uniformity Conjecture.*

In this talk, I will discuss recent results which aim to classify the elliptic curves producing sporadic points on $X_1(N)$*, spanning projects which are joint with David Gill, Jeremy Rouse, Lori Watson, and Filip Najman. In keeping with the theme of the workshop, I will highlight central parts of the argument which involve finding all rational points on certain "entanglementmodular curves. Several directions for future investigation will be presented.*

Michael Stoll: **Rational points on and BSD for a curve of genus 4**

I will explain how to determine the set of rational points on a certain non-hyperelliptic curve C *of genus* 4 *over* $\mathbb{Q}$*. The main ingredient is the determination of the Mordell-Weil group. I will then use this to also numerically verify strong BSD (up to odd squares) for the Jacobian of* C*.*