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Finite Descent Obstructions and Rational Points

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Motivation

k is always a number field.

Question.
Given a smooth projective curve C/k,
can we efficiently decide whether $C(k) = \emptyset$ or not?

- If $C(k) \neq \emptyset$, we can find a point $\implies$ OK.
- If $C(\mathbb{A}_k) = \emptyset$, then $C(k) = \emptyset$ $\implies$ OK.
- If $C(\mathbb{A}_k) \neq \emptyset$, but apparently $C(k) = \emptyset$,
  we can try *descent*.

Descent

Let $\pi : D \longrightarrow C$ be finite étale, geometrically Galois
(more precisely: a C-torsor under a finite k-group scheme G).

Then we have the *twists* $\pi_\xi : D_\xi \longrightarrow C$ for $\xi \in H^1(k, G)$.

Theorem.

- $C(k) = \bigcup_{\xi \in H^1(k,G)} \pi_\xi(D_\xi(k))$.
- $\mathrm{Sel}^\pi(k, C) := \{\xi \in H^1(k, G) : D_\xi(\mathbb{A}_k) \neq \emptyset\}$ is finite (and computable).

(Fermat, Chevalley-Weil, . . .)

If we find $\mathrm{Sel}^\pi(k, C) = \emptyset$, then $C(k) = \emptyset \quad \Longrightarrow$ OK.

Questions.

- Does this always work to prove $C(k) = \emptyset$?
- How much information on $C(k) \subset C(\mathbb{A}_k)$ can we get in this way?

A Definition

Note.
At infinite places, we only get information on connected components.

Therefore: for X/k smooth projective, set

$$X(\mathbb{A}_k)_\bullet = \prod_{v \nmid \infty} X(k_v) \times \prod_{v \mid \infty} \pi_0(X(k_v))$$

Definition.

- For $\pi : Y \longrightarrow X$ torsor under G set $X(\mathbb{A}_k)_\bullet^\pi = \bigcup_{\xi \in H^1(k,G)} \pi_\xi(Y_\xi(\mathbb{A}_k)_\bullet)$.
- $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \bigcap_{\pi} X(\mathbb{A}_k)_\bullet^\pi$.
- $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \bigcap_{\pi \text{ abelian}} X(\mathbb{A}_k)_\bullet^\pi$.

First Properties

With $\overline{X(k)}$ the topological closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet$:

$$X(k) \subset \overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet$$

- $f : X \longrightarrow Y$ morphism, then

$$f(X(\mathbb{A}_k)_\bullet^{\text{f-cov/f-ab}}) \subset Y(\mathbb{A}_k)_\bullet^{\text{f-cov/f-ab}}$$

- K/k finite extension, then

$$i_{K/k}(X(\mathbb{A}_k)_\bullet^{\text{f-cov/f-ab}}) \subset X(\mathbb{A}_K)_\bullet^{\text{f-cov/f-ab}}$$

Abelian Varieties

Let A/k be an abelian variety.

Every (geom. connected) abelian covering of A is a quotient of $A \xrightarrow{\cdot n} A$ for some n.

So we consider Selmer groups:

$$\begin{array}{ccc} \mathrm{Sel}^{(n)}(k, A) & \subset & H^1(k, A[n]) \\ \downarrow & & \downarrow \\ A(\mathbb{A}_k)/nA(\mathbb{A}_k) & \hookrightarrow & \prod_v H^1(k_v, A[n]) \end{array}$$

For $n \mid N$, have commutative diagram

$$\begin{array}{ccc} \mathrm{Sel}^{(N)}(k, A) & \longrightarrow & \mathrm{Sel}^{(n)}(k, A) \\ \downarrow & & \downarrow \\ A(\mathbb{A}_k)/NA(\mathbb{A}_k) & \twoheadrightarrow & A(\mathbb{A}_k)/nA(\mathbb{A}_k) \end{array}$$

The Pro-Finite Selmer Group

Pass to the limit to obtain the *pro-finite Selmer group*

$$\widehat{\mathsf{Sel}}(k, A) = \varprojlim \mathsf{Sel}^{(n)}(k, A)$$

The exact sequences

$$0 \longrightarrow A(k)/nA(k) \longrightarrow \mathsf{Sel}^{(n)}(k, A) \longrightarrow Ш(k, A)[n] \longrightarrow 0$$

piece together to give

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{A(k)} & \longrightarrow & \widehat{\mathsf{Sel}}(k, A) & \longrightarrow & TШ(k, A) & \longrightarrow & 0 \\
 & & \uparrow & & \downarrow & & & & \\
 & & A(k) & \longrightarrow & \varprojlim A(\mathbb{A}_k)/nA(\mathbb{A}_k) & = & A(\mathbb{A}_k)_\bullet & &
\end{array}$$

- $TШ(k, A) = 0$ if and only if $Ш(k, A)_{\mathsf{div}} = 0$.
- $A(\mathbb{A}_k)_\bullet^{\text{f-ab}} =$ image of $\widehat{\mathsf{Sel}}(k, A)$ in $A(\mathbb{A}_k)_\bullet$.

A Theorem

Theorem.
Let $Z \subset A$ be a finite subscheme and

$$\widehat{\mathsf{Sel}}(k, A) \ni \hat{P} \longmapsto (P_v)_v \in A(\mathbb{A}_k)_\bullet$$

such that $P_v \in Z(k_v)$ for v in a set of density 1.
Then $\hat{P} \in Z(k)$.

Consequences.

- $\widehat{\mathsf{Sel}}(k, A)$ injects into $A(\mathbb{A}_k)_\bullet$, identifying $\widehat{A(k)}$ with $\overline{A(k)}$.
- $A(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \widehat{\mathsf{Sel}}(k, A)$, and $Z(\mathbb{A}_k)_\bullet \cap A(\mathbb{A}_k)_\bullet^{\text{f-ab}} = Z(k)$.
- We even have $\widehat{\mathsf{Sel}}(k, A) \hookrightarrow \prod_{v \in S} A(\mathbb{F}_v)$ if S has density 1.

Obstruction on Abelian Varieties and PHSs

With the identifications just found, we have an exact sequence

$$0 \longrightarrow \overline{A(k)} \longrightarrow A(\mathbb{A}_k)_\bullet^{\textsf{f-ab}} \longrightarrow T\text{Ш}(k, A) \longrightarrow 0$$

This implies

- $\overline{A(k)} = A(\mathbb{A}_k)_\bullet^{\textsf{f-ab}} \iff \text{Ш}(k, A)_{\textsf{div}} = 0$.
- $A(k) = A(\mathbb{A}_k)_\bullet^{\textsf{f-ab}} \iff \text{Ш}(k, A)_{\textsf{div}} = 0 \quad \text{and} \quad \operatorname{rank} A(k) = 0$ (e.g., $A/\mathbb{Q}$ modular of analytic rank zero (Kolyvagin-Logachev)).

If X is a principal homogeneous space for A with $X(k) = \emptyset$, $X(\mathbb{A}_k)_\bullet \neq \emptyset$, then X represents $0 \neq \xi \in \text{Ш}(k, A)$, and

- $\emptyset = X(k) = X(\mathbb{A}_k)_\bullet^{\textsf{f-ab}} \iff \xi \notin \text{Ш}(k, A)_{\textsf{div}}$.

Proof of Theorem (Sketch)

- Let $k_N = k(A[N])$ be the N-division field.
- Result of Serre's on image of Galois in $\mathrm{Aut}(A_{\mathrm{tors}})$ implies

$$\exists m \forall N : m \text{ kills } \ker\Big(\mathrm{Sel}^{(N)}(k, A) \to \mathrm{Sel}^{(N)}(k_N, A)\Big)$$

- From this: if $Q \in \mathrm{Sel}^{(N)}(k, A)$, $\mathrm{ord}(mQ) = n$,
  then the set of places v such that
  (i) v splits completely in k_N/k, and (ii) $Q_v = 0 \in H^1(k_v, A[N])$
  has density $\leq 1/(n[k_N : k])$ (Chebotarev).
- Assume that $Z(\bar{k}) = Z(k)$ (make field extension in general case).
- If $\hat{P} \notin Z(k) + A(k)_{\mathrm{tors}}$, then we find set of v of positive density
  such that $P_v \neq Q$ for all $Q \in Z(k)$, contradiction.
  (Note $\widehat{\mathrm{Sel}}(k, A)_{\mathrm{tors}} = A(k)_{\mathrm{tors}}$.)
- So $\hat{P} \in Z(k) + A(k)_{\mathrm{tors}} \subset A(k) \hookrightarrow A(k_v)$, and $\hat{P} = P_v \in Z(k)$
  (pick suitable v).

Curves

Let C/k be a curve of genus g with Jacobian J.

- $g = 0$: Hasse Principle.
- $g = 1$: C is abelian variety or PHS.
- $g \geq 2$: $C(k)$ is finite, so $C(k) = \overline{C(k)}$.

By Geometric Class Field Theory, all abelian coverings of C come from J.

- If $\mathrm{Pic}^1_C(k) = \emptyset$ and $[\mathrm{Pic}^1_C] \notin Ш(k, J)_{\mathrm{div}}$, then $C(k) = C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \emptyset$.
- If $\mathrm{Pic}^1_C(k) \neq \emptyset$, then $\exists \iota : C \hookrightarrow J$, and $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \iota^{-1}(\widehat{\mathrm{Sel}}(k, J))$.

  If $Ш(k, J)_{\mathrm{div}} = 0$, and we identify C with $\iota(C)$:

  $$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \overline{J(k)} \cap C(\mathbb{A}_k)_\bullet$$

  (this can be used for computations).

Note: $Ш(k, J)_{\mathrm{div}} = 0$ and $J(k)$ finite $\implies$ $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(k)$.

Adelic Mordell-Lang?

Question.
Is there an *Adelic Mordell-Lang Conjecture*?

E.g., if $X \subset A$ not containing a nontrivial subabelian variety, then

$$\exists Z \subset X \text{ finite} : X(\mathbb{A}_k)_\bullet \cap \widehat{\mathsf{Sel}}(k, A) \subset Z(\mathbb{A}_k)_\bullet$$

Remark.
True with $\overline{A(k)}$ instead of $\widehat{\mathsf{Sel}}(k, A)$ if
k is global function field, A ordinary, X not defined over k^p (Voloch).

If the above is true, then for a curve $C \subset J$, we have

$$C(k) \subset C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(\mathbb{A}_k)_\bullet \cap \widehat{\mathsf{Sel}}(k, J) \subset Z(\mathbb{A}_k)_\bullet \cap \widehat{\mathsf{Sel}}(k, J) = Z(k) \subset C(k)$$

and so $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(k)$.

Main Conjecture

Conjecture.
If C/k is a smooth projective curve of genus ≥ 2, then $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(k)$.

- K/k finite extension, $C(\mathbb{A}_K)_\bullet^{\text{f-ab}} = C(K)$, then $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(k)$.
- $C \longrightarrow X$ non-constant, $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(k)$, then $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(k)$.
  (Use Theorem $\Longrightarrow Z(\mathbb{A}_k)_\bullet \cap C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = Z(k)$ for $Z \subset C$ finite.)
- $C = X_0(N), X_1(N), X(N)$ satisfy $C(\mathbb{A}_\mathbb{Q})_\bullet^{\text{f-ab}} = C(\mathbb{Q})$ if genus ≥ 1.
- Many more examples.
- $C : y^2 = f(x)$, $g = 2$, coeffs of f in $\{-3, \ldots, 3\}$,
  then $C(\mathbb{A}_\mathbb{Q})_\bullet^{\text{f-ab}} = \emptyset$ whenever $C(\mathbb{Q}) = \emptyset$. (Bruin-Stoll)
  (need to assume $Ш(k, J)_{\text{div}} = 0$ for 1492 cases, BSD for 42 cases)
- Poonen has a stronger conjecture when $C(k) = \emptyset$,
  supported by heuristic arguments.

Consequences

- We can effectively decide if $C(k) = \emptyset$ or not.

- The Brauer-Manin Obstruction is the only obstruction against rational points on C and against weak approximation in $C(\mathbb{A}_k)_\bullet$.

Comparison With Brauer-Manin Obstruction

X/k smooth projective, geometrically irreducible.

Recall

$$\mathrm{Br}_1(X) = \ker\Big(\mathrm{Br}(X) \to \mathrm{Br}(X \times_k \bar{k})\Big) \quad \text{and} \quad \mathrm{Br}_1(X) \twoheadrightarrow H^1(k, \mathrm{Pic}_X)$$

Let $\mathrm{Br}_{1/2}(X) \subset \mathrm{Br}_1(X)$ be the preimage of the image of $H^1(k, \mathrm{Pic}^0_X)$.

Then

$$X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}$$

($X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$: Colliot-Thélène & Sansuc, Skorobogatov)

If $X = C$ is a curve, then $\mathrm{Br}(C) = \mathrm{Br}_1(C) = \mathrm{Br}_{1/2}(C)$, and so

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$$