How many rational points can a curve have?*

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*title borrowed from Caporaso–Harris–Mazur 1995

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- $g \geq 2$: $C(K)$ is finite (Faltings (1983), conjectured by Mordell (1922)).

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Stoll (2008): $B(2, 1) \geq 642$

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($\star$) If $\Gamma \subset J(\mathbb{C})$ is of finite rank*, then $\mathrm{im}(\iota) \cap \Gamma$ is finite.

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So Weil + ($\star$) $\implies$ $\#C(K) < \infty$.

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Note that $c_2(g) \to 1$ as $g \to \infty$; this was expected.

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Assuming that $\mathrm{rk}\,A(\mathbb{Q}) \leq B'(g)$
for all abelian varieties A of dimension g over $\mathbb{Q}$,
we obtain the uniform bound $\#C(K) \leq c_1(g) \cdot c_2(g)^{B'(g[K:\mathbb{Q}])}$.

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The constants involved are explicit, and small (< 100) for $K = \mathbb{Q}$.
Note, however, that these results are not uniform in K.

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These densities should in fact be zero.

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  (A abelian variety over $\mathbb{Q}$.) This is ≥ 29 (Elkies–Klagsbrun).
- What can we say about $\limsup\limits_{g(C) \to \infty} \frac{\#C(\mathbb{Q})}{g(C)}$ and/or $\sup\limits_{C} \frac{\#C(\mathbb{Q})}{g(C)}$?*
  (C nice curve over $\mathbb{Q}$ with $g(C) \ge 2$.)
  They are ≥ 8 and ≥ 321, respectively.

*this was already considered by Caporaso, Harris and Mazur (1995)

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- The explicit bounds we now have are still much larger than what we expect to be true.
- There are better bounds that apply in restricted settings.
- A very interesting (but certainly extremely hard) question is whether certain quantities ($\operatorname{rk} A(\mathbb{Q})$, $\#C(\mathbb{Q})$) grow linearly or superlinearly with the dimension/genus.

Thank You!