Simultaneous Torsion in the Legendre Family of Elliptic Curves

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Introduction

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For $\alpha \in \mathbb{C} \setminus \{0, 1\}$, let $P_\lambda(\alpha) \in E_\lambda$ be a point with x-coordinate α and define

$$T(\alpha) = \{\lambda \in \mathbb{C} \setminus \{0, 1\} : P_\lambda(\alpha) \in E_\lambda(\mathbb{C}) \text{ is torsion}\}.$$

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consisting of elements algebraic over $\mathbb{Q}(\alpha)$.

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Then $T(\alpha)$ is a countably infinite set
consisting of elements algebraic over $\mathbb{Q}(\alpha)$.

Now consider $\alpha, \beta \in \mathbb{C} \setminus \{0, 1\}$ with $\alpha \neq \beta$ and set $T(\alpha, \beta) = T(\alpha) \cap T(\beta)$.

Question.
What can we say about $T(\alpha, \beta)$?

Known Results

There are three cases:

- α and β are algebraic.
- $\mathrm{trdeg}_{\mathbb{Q}}\big(\mathbb{Q}(\alpha,\beta)\big) = 1$.
- α and β are algebraically independent. Then $T(\alpha,\beta) = \emptyset$.

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Masser and Zannier showed that $T(2,3)$ is finite
and then proved the following more general result.

Theorem (Masser and Zannier).
$T(\alpha,\beta)$ is always finite; when $\text{trdeg}_{\mathbb{Q}}\big(\mathbb{Q}(\alpha,\beta)\big) = 1$, this is effective.

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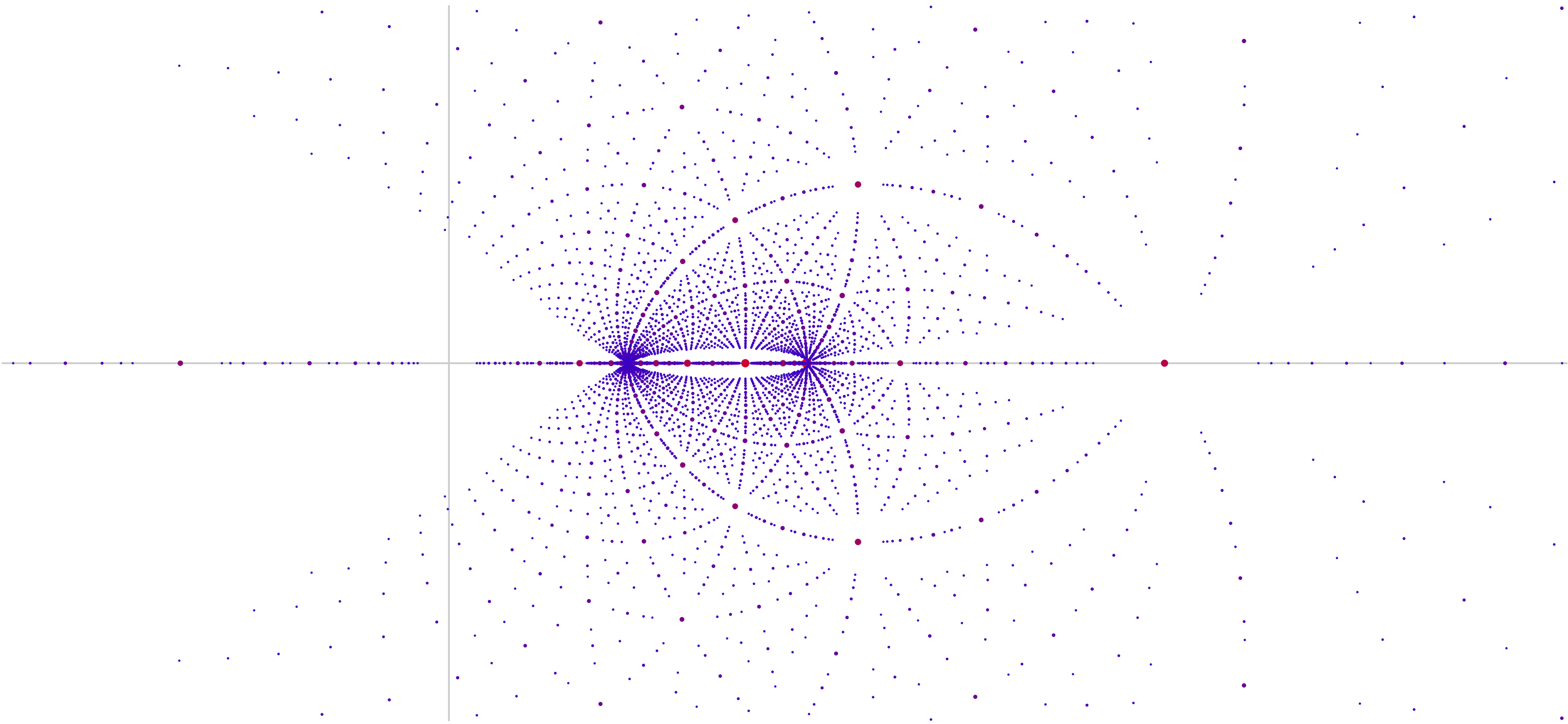
Goals of this talk:
(1) Get effectivity for some algebraic α,β.
(2) Get optimal result for transcendence degree 1.
(3) Use this to get more information on the algebraic case.

Structure of $T(\alpha)$

In $\mathbb{C}$, $T(\alpha)$ is all over the place,
reflecting the fact that E_{tors} is dense in $E(\mathbb{C})$:



This shows $T_{40}(2)$, where $T_n(\alpha) = \{\lambda \in T(\alpha) : P_\lambda(\alpha) \in E_\lambda \text{ has order} \leq n\}$.

Aside

DeMarco, Wang and Ye show
that there is actually a limiting distribution μ_α
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This gives an alternative proof of the Masser-Zannier result.

Structure of $T(\alpha)$, p-adically

Fix a prime p.
In contrast to the situation over $\mathbb{C}$, E_{tors} is discrete in $E(\mathbb{C}_p)$.
This translates into $T(\alpha)$ being discrete in $\mathbb{C}_p \setminus \{0, 1\}$.

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Since $T(\alpha)$ moves continuously with α,
we can show that $T(\alpha, \beta)$ is empty if α and β are p-adically close:

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Proposition.
Let $\alpha, \beta \in \mathbb{C}_p$ with $0 < |\alpha(\alpha - 1)|_p \leq 1$ and $0 < |\beta - \alpha|_p < |\alpha(\alpha - 1)|_p \, |p|_p^{2/(p-1)}$.
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We also get that $T(\alpha, \beta) = \emptyset$ when $|\alpha|_p < |p|_p^{2/(p-1)}$ and $|\beta - 1|_p < |p|_p^{2/(p-1)}$.

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If we can show that $T(\alpha) \subset \mathbb{C}_p$ is sufficiently localized,
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Easy Lemma.

For $\alpha, \lambda \in \mathbb{C}_p \setminus \{0, 1\}$ the following are equivalent:

- $\lambda \in T(\alpha)$.
- $\lambda = \alpha$, or $\psi_n(\lambda, \alpha) = 0$ for some $n \geq 3$, where $\psi_n(\lambda, x)$ is the nth division polynomial of E_λ.
- α is preperiodic for the Lattès map $f_\lambda \colon x \longmapsto \dfrac{(x^2 - \lambda)^2}{4x(x-1)(x-\lambda)}$ on $\mathbb{P}^1$.

  (This point of view was used by Mavraki.)

2-adic Localization

We look specifically at $p = 2$. $|\cdot|$ denotes the 2-adic absolute value.

It is easy to see that $T(1/\alpha) = \{1/\lambda : \lambda \in T(\alpha)\}$, so we can assume that $|\alpha| \leq 1$.
Then for all $\lambda \in T(\alpha)$, we have $|\lambda| \leq 1$ as well
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So if $\lambda \in T(\alpha)$, we must have that $\lambda = \alpha$ ($\iff$ $f_\lambda(\alpha) = \infty$) or $|f_\lambda(\alpha)| \leq 1$.
The latter means $|\lambda - \alpha^2|^2 \leq |4\alpha(\alpha - 1)(\alpha - \lambda)| \leq |4|$, which says that

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Corollary. $T(2, 3) = \emptyset$.

A Slightly More Precise Result

Note that we have

$$\lambda \in T(\alpha) \iff f_\lambda(\alpha) \in \{0, 1, \lambda, \infty\} \quad \text{or} \quad \lambda \in T(f_\lambda(\alpha))\,.$$

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The first condition is

$$\lambda \in S(\alpha) := \left\{\alpha, \alpha^2, \alpha(2-\alpha), \tfrac{\alpha^2}{2\alpha-1}\right\}.$$

We can easily show that for $|\alpha| \leq 1$ (similarly for $|\alpha| > 1$),

$$T(f_\lambda(\alpha)) \subset R(\alpha) := \{\alpha^2 + 2u\alpha(1-\alpha) : u \in \mathbb{C}_2, |u^2 - \alpha| < 1\}.$$

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So if $R(\alpha) \cap R(\beta) = \emptyset$, then we can determine $T(\alpha, \beta)$:

$$T(\alpha, \beta) \subset S(\alpha) \cup S(\beta).$$

This will be the case when α and β are 2-adically sufficiently distinct.

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(4) $\mathsf{T}(\omega,\omega^2) = \{\omega,\omega^2\}$, where ω is a cube root of unity.

Let $\boldsymbol{\mu}$ be the set of all roots of unity.
Then $\#(\mathsf{T}(\alpha) \cap \boldsymbol{\mu}) \leq 3$ for all α, and

$$\#(\mathsf{T}(\alpha) \cap \boldsymbol{\mu}) = 3 \iff \alpha \in \boldsymbol{\mu} \quad \text{and} \quad \mathrm{ord}(\alpha) \in \{3,6,12\}.$$

Transcendence Degree 1

Assume that $\mathrm{trdeg}_{\mathbb{Q}}\big(\mathbb{Q}(\alpha, \beta)\big) = 1$
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$$\big(\lambda = \alpha \text{ or } \exists n \geq 3\colon \psi_n(\lambda,\alpha) = 0\big) \quad \text{and} \quad \big(\lambda = \beta \text{ or } \exists n \geq 3\colon \psi_n(\lambda,\beta) = 0\big).$$

Eliminating λ, we see that F divides
$\psi_n(a,b)$ or $\psi_n(b,a)$ or $R_n(a,b) := \mathrm{Res}_t(\psi_n(t,a),\psi_n(t,b))/(a-b)^{\deg_t \psi_n(t,x)}$,
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for some $n \geq 3$.

Proposition 1.
For all $n \geq 3$, the polynomial $\psi_n(a,b)\psi_n(b,a)R_n(a,b)$ is squarefree in $\mathbb{Q}[a,b]$.

Sketch of proof. Write the possible b near $a = 0$ as Puiseux series in a (using Tate parameterization) and check that they are distinct.

Result

Let, for $n \geq 3$, C_n be the curve in $\mathbb{P}^1_a \times \mathbb{P}^1_b$ given by

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This implies that each component of C corresponds
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Proposition 2.
Let $\alpha, \beta \in \mathbb{C} \setminus \{0, 1\}$ with $\alpha \neq \beta$. Then
$\#T(\alpha, \beta) \leq$ the number of branches of C passing through (α, β).

Consequences

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If $F = 0$ describes a component of C, we can bound n in terms of $\deg F$.
This gives effectivity in the $\operatorname{trdeg} = 1$ case.
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(Masser and Zannier show $\#T(\alpha, \beta) \leq 6(12 \deg F)^{32}$ when trdeg $= 1$.)

Computations

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Based on this,
we computed all singularities on components with $(\deg_{ab} F)^2 \leq 384$
and all intersections of components with $(\deg_{ab} F_1)(\deg_{ab} F_2) \leq 384$.
We then computed $T_{50}(\alpha, \beta) = T_{50}(\alpha) \cap T_{50}(\beta)$ for these points (α, β),
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558 of these have $\#T_{50}(\alpha,\beta) \geq 3$ (with all torsion orders ≤ 18),
15 of these have $\#T_{50}(\alpha,\beta) \geq 4$,
and 3 of these have $\#T_{50}(\alpha,\beta) = 5$; a representative is $(i,-i)$ with

$$T_{100}(i,-i) = \{-1, 3 \pm 2\sqrt{2}, \tfrac{1}{3} \pm \tfrac{2}{3}\sqrt{-2}\}.$$

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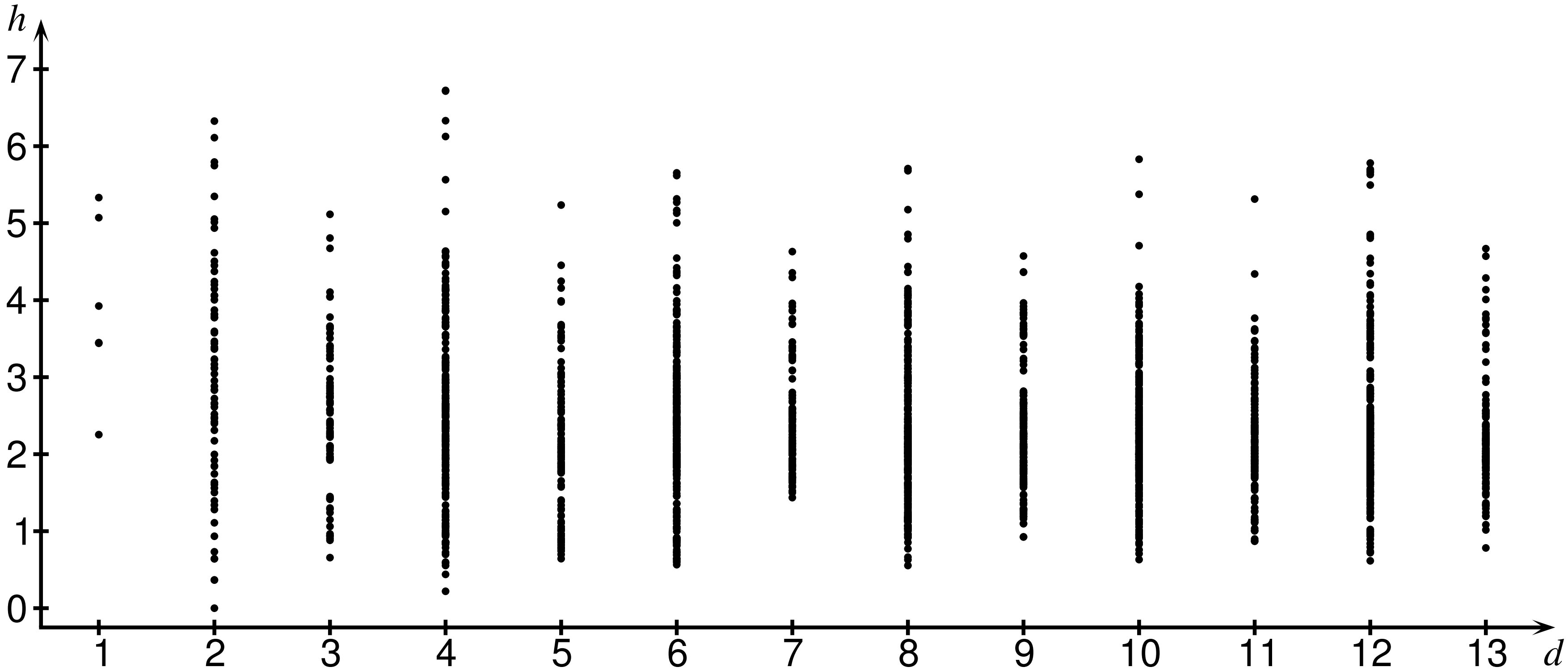
Conjecture 5 (Bounded degree).

There is a uniform bound for $\left[\mathbb{Q}(\alpha,\beta,\lambda) : \mathbb{Q}(\alpha,\beta)\right]$ when $\lambda \in T(\alpha,\beta)$.
The bound might even by 2.

Conjecture 5 would imply effectivity of $T(\alpha,\beta)$.

Heights

This shows the (symmetrized) heights h of pairs (α, β) with $\#T(\alpha, \beta) \geq 2$, ordered according to the degree d of $\mathbb{Q}(\alpha, \beta)$.



Thank You!