THE SURFACE PARAMETRIZING CUBOIDS

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ABSTRACT. We study the surface $\bar{S}$ parametrizing cuboids: it is defined by the equations relating the sides, face diagonals and long diagonal of a rectangular box. It is an open problem whether a 'rational box' exists, i.e., a rectangular box all of whose sides, face diagonals and long diagonal have (positive) rational length. The question is equivalent to the existence of nontrivial rational points on $\bar{S}$.

Let S be the minimal desingularization of $\bar{S}$ (which has 48 isolated singular points). The main result of this paper is the explicit determination of the Picard group of S, including its structure as a Galois module over $\mathbb{Q}$. The main ingredient for showing that the known subgroup is actually the full Picard group is the use of the combined action of the Galois group and the geometric automorphism group of S (which we also determine) on the Picard group. This reduces the proof to checking that the hyperplane section is not divisible by 2 in the Picard group.

1. INTRODUCTION

Let $\mathbb{P}^6$ be the projective space over $\mathbb{Q}$ with homogeneous coordinates a_1, a_2, a_3, b_1, b_2, b_3, c; let $\bar{S}$ be the surface in $\mathbb{P}^6$ defined by

$$(1)\qquad \begin{cases} a_1^2 + b_1^2 = c^2 \\ a_2^2 + b_2^2 = c^2 \\ a_3^2 + b_3^2 = c^2 \\ a_1^2 + a_2^2 + a_3^2 = c^2 \end{cases}$$

and note that the equations in (1) are equivalent to the equations

$$(2)\qquad \begin{cases} a_1^2 + a_2^2 = b_3^2 \\ a_1^2 + a_3^2 = b_2^2 \\ a_2^2 + a_3^2 = b_1^2 \\ a_1^2 + a_2^2 + a_3^2 = c^2. \end{cases}$$

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These equations encode the relations between the three sides a_1, a_2, a_3, the three face diagonals b_1, b_2, b_3 and the long diagonal c of a three-dimensional rectangular box.

The interest in this surface comes from a famous open problem:

Does there exist a 'rational box'?

A rational box is a (non-degenerate) rectangular box all of whose sides, face diagonals and long diagonals have rational length. The existence of a rational box is therefore equivalent to the existence of a rational point on $\bar{S}$ with $a_1a_2a_3 \neq 0$. See [vL] for a summary of the literature on this problem.

In this paper, we hope to make progress toward a better understanding of the rational box surface by proving some results on its geometry. This extends results obtained by van Luijk in his undergraduate thesis [vL].

The surface $\bar{S}$ has 48 isolated A_1 singularities. We let S denote the minimal desingularization of $\bar{S}$. Then we show the following.

Theorem 1. $\mathrm{Aut}_{\overline{\mathbb{Q}}}(S) = \mathrm{Aut}_{\overline{\mathbb{Q}}}(\bar{S})$ *is an explicitly given group of order 1536. Its action on* $\bar{S}$ *extends to a linear action on the ambient* $\mathbb{P}^6$.

See Section 2. The group was already known to van Luijk; we prove here that it is the full automorphism group.

Theorem 2. *The geometric Picard group of* S *has maximal rank*

$$\operatorname{rank} \operatorname{Pic} S_{\overline{\mathbb{Q}}} = \dim \mathrm{H}^{1,1}(S(\mathbb{C})) = 64.$$

It is generated by an explicitly known set of curves on $\bar{S}$ *together with the exceptional divisors. The discriminant of the intersection pairing on the Picard group is* -2^{28}.

See Section 3. It is not hard to show that the geometric Picard rank is 64, since one easily finds enough curves to generate a group of that rank. These curves are already in [vL]. The hard part is to show that the known curves generate the full Picard group and not a proper subgroup of finite index. Since 2 is the only prime number dividing the discriminant of the known subgroup, it remains to show that no primitive element of the known subgroup is divisible by 2. We use the known action of the automorphism group together with the action of the absolute Galois group of $\mathbb{Q}$ to reduce the proof of saturation to the statement that the single element corresponding to the hyperplane section is not divisible by 2. This claim is then fairly easily established.

This technique, especially the arguments in the proof of Theorem 7, may be helpful in similar situations, when one has a fairly large group acting on the Picard group. Indeed, A. Várilly-Alvarado and B. Viray [VV] use this technique to compute the Picard groups of various Enriques surfaces.

2. The Automorphism Group

In this section, we determine the automorphism group of S and $\bar{S}$. We begin with some basic geometric properties of $\bar{S}$.

Lemma 3. *The scheme $\bar{S}$ is a geometrically integral complete intersection of dimension two and multidegree $(2,2,2,2)$ in $\mathbb{P}^6$ with 48 isolated A_1 singularities.*

Proof. Every irreducible component of $\bar{S}$ has dimension at least two, since $\bar{S}$ is defined by four equations. If $\bar{S}$ had a component of dimension at least three, then the intersection of $\bar{S}$ with any hyperplane would have a component of dimension at least two. Let C be the hyperplane defined by the vanishing of c. The scheme $\bar{S} \cap C$ is the union of eight smooth conics defined over the field $\mathbb{Q}(i)$ by

$$c = 0, \quad b_1 = \varepsilon_1 i a_1, \quad b_2 = \varepsilon_2 i a_2, \quad b_3 = \varepsilon_3 i a_3, \quad a_1^2 + a_2^2 + a_3^2 = 0$$

for $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$. In particular it is pure of dimension one, so that every irreducible component of $\bar{S}$ has dimension two; note also that $\bar{S} \cap C$ is reduced. It follows that $\bar{S}$ is the complete intersection of the equations in (1).

Thus $\bar{S}$ is Cohen-Macaulay of dimension two, and to prove that it is integral it suffices to show that the singular locus of $\bar{S}$ has codimension at least two. Since $\bar{S} \cap C$ is reduced, no component of dimension one of the singular locus of $\bar{S}$ is contained in C. Let $p = [\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma]$ be a point in $\bar{S} \setminus C$. Examining the equations of $\bar{S}$ we see immediately that if $\alpha_1\alpha_2\alpha_3 \neq 0$ or $\beta_1\beta_2\beta_3 \neq 0$, then the rank of the jacobian of the equations at p is four and such points are therefore smooth. We conclude at once that the singular points of $\bar{S}$ are the points for which all three coordinates appearing in one of the six rank three quadrics in (1) or (2) vanish; in particular, the surface $\bar{S}$ has only finitely many singular points. The fact that the singular points are 48 and that they are of type A_1 is immediate from the equations. □

As a corollary, we deduce that $\bar{S}$ is Cohen-Macaulay, Gorenstein, reduced, normal and projectively normal. By the adjunction formula, the canonical sheaf on $\bar{S}$ is the sheaf $\mathcal{O}_{\bar{S}}(1)$. Since $\bar{S}$ is projectively normal we deduce also that $p_g(\bar{S}) = 7$; it is an easy calculation to see that $\chi(\bar{S}, \mathcal{O}_{\bar{S}}) = 8$. Let $b \colon S \to \bar{S}$ be the blow-up of $\bar{S}$ at its 48 singular points; thus S is smooth and it is the minimal desingularization of $\bar{S}$. Denote by K_S a canonical divisor of S; we have $\mathcal{O}_S(K_S) \simeq b^*\mathcal{O}_{\bar{S}}(1)$ and $(K_S)^2 = 16$. Since the singularities of $\bar{S}$ are rational double points, we have $\chi(S, \mathcal{O}_S) = \chi(\bar{S}, \mathcal{O}_{\bar{S}}) = 8$ and also $p_g(S) = 7$ and $q(S) = 0$. Using Noether's formula we finally deduce that the Hodge diamond of S is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 7 & & 64 & & 7 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

It follows from the above that the rational map associated to the canonical divisor K_S on S is a morphism and that it is the contraction of the 48 exceptional curves of b, followed by the inclusion of $\bar{S}$ into $\mathbb{P}^6$. Therefore the canonical divisor on S is big and nef, so that S is a minimal surface of general type and $\bar{S}$ is its canonical model: there are no curves on S with negative intersection with the canonical divisor, and, in particular, there are no (-1)-curves on S. Moreover, the 48 exceptional curves of b are the only (-2)-curves on S. Since the morphism $S \to \mathbb{P}^6$ is the morphism associated to the canonical divisor, the automorphism groups of S and $\bar{S}$ coincide; denote this group by G. The group G is naturally identified with the subgroup of $\operatorname{Aut}(\mathbb{P}^6)$ preserving the subscheme $\bar{S}$, since the canonical divisor class of S is G-invariant.

The symmetric group $\mathfrak{S}_3$ on $\{1,2,3\}$ acts on $\mathbb{P}^6$ and $\bar{S}$ by permuting simultaneously the indices of a_1, a_2, a_3 and b_1, b_2, b_3 and fixing c. Note that also the linear automorphism of order two

$$\sigma : \begin{cases} a_1 \mapsto a_1 & b_1 \mapsto -ib_2 \\ a_2 \mapsto a_2 & b_2 \mapsto ib_1 \\ a_3 \mapsto -ic & b_3 \mapsto b_3 \\ c \mapsto ia_3 & \end{cases}$$

of $\mathbb{P}^6$ preserves $\bar{S}$ and therefore induces an automorphism of S. Let $G' \subset G$ be the subgroup generated by $\mathfrak{S}_3$, σ and all the sign changes of the variables.

Proposition 4. *The group $G = \operatorname{Aut}(S)$ is equal to G'.*

Proof. First, we show that the rank three quadrics vanishing on $\bar{S}$ are the six rank three quadrics appearing in (1) and (2). For $j \in \{1,2,3\}$, let

$$q_j := a_j^2 + b_j^2 - c^2 \quad \text{and} \quad r_j := a_1^2 + a_2^2 + a_3^2 - a_j^2 - b_j^2,$$

and let $Q_j := V(q_j)$ and $R_j := V(r_j)$. Let

$$q_4 := a_1^2 + a_2^2 + a_3^2 - c^2,$$

and let q be an equation of a quadric vanishing on $\bar{S}$. Since the ideal of $\bar{S}$ is generated by the equations in (1), we have $q = \lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3 + \lambda_4 q_4$, for some $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \overline{\mathbb{Q}}$.

If at least two of the coefficients $\lambda_1, \lambda_2, \lambda_3$ are non-zero, then we easily see that q has rank at least four. Thus at most one of the coefficients $\lambda_1, \lambda_2, \lambda_3$ is non-zero, and we conclude that the rank three quadrics vanishing on $\bar{S}$ are $Q_1, Q_2, Q_3, R_1, R_2, R_3$. It follows that the set $\mathcal{Q} := \{Q_1, Q_2, Q_3, R_1, R_2, R_3\}$ is fixed by the induced action of G.

Second, we show that G' acts transitively on $\mathcal{Q}$. This is immediate: the group $\mathfrak{S}_3$ acts transitively on $\{Q_1, Q_2, Q_3\}$ and $\{R_1, R_2, R_3\}$, while σ acts transitively on $\{Q_1, R_2\}$ and $\{Q_2, R_3\}$ (and it fixes Q_3 and R_1).

Third, we show that the stabilizer of Q_1 in G is the same as the stabilizer of Q_1 in G'; from this the result follows since G and G' act transitively on $\mathcal{Q}$. Let $\tau \in G$ be an automorphism of $\mathbb{P}^6$ stabilizing $\bar{S}$ and Q_1. In particular τ stabilizes Q_1^{sing}, the singular locus of Q_1, and $Q_1^{\text{sing}} \cap \bar{S}$. The (underlying set of the) intersection $Q_1^{\text{sing}} \cap \bar{S}$ is the set of eight points $\big\{[0, 1, \varepsilon_1, 0, \varepsilon_2 i, \varepsilon_3 i, 0] : \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}\big\}$. Clearly the actions of the stabilizers in G and G' are transitive on this set, since both groups contain arbitrary sign changes of the variables, and if an automorphism of $V(a_1, b_1, c) \simeq \mathbb{P}^3$ fixes the eight points above, then it is the identity (the points for which $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$ are essentially the characters of $(\mathbb{Z}/2\mathbb{Z})^2$, and therefore the corresponding points span $\mathbb{P}^3$). Thus, we may assume that τ acts as the identity on $V(a_1, b_1, c) \subset \mathbb{P}^6$. Hence τ fixes the variable c, up to a sign, since it fixes a_2, b_2; similarly it fixes a_1 and b_1 up to signs, since it fixes a_2, b_3 and a_2, a_3 respectively, and we conclude that $G' = \operatorname{Aut}(\bar{S}) = \operatorname{Aut}(S)$. □

To compute the size of the group G, note that each element of $G = G'$ induces a permutation of the set $\{a_1^2, a_2^2, a_3^2, -c^2\}$ and every permutation is obtained. This gives a surjective group homomorphism $G \to \mathfrak{S}_4$. The subgroup $\langle \mathfrak{S}_3, \sigma \rangle$ permutes $\{a_1, a_2, a_3, -ic\}$. One can check that the kernel of $\langle \mathfrak{S}_3, \sigma \rangle \to \mathfrak{S}_4$ consists of maps that fix b_1^2, b_2^2, b_3^2. So the kernel of $G \to \mathfrak{S}_4$ consists exactly of the automorphisms that change the signs of a subset of the variables. This shows that

$$\#G = 2^{7-1} \# \mathfrak{S}_4 = 64 \cdot 24 = 1536.$$

The group G' is described in [vL, p. 25] as a subgroup of the automorphism group of $\bar{S}$. The new statement here is that it is already the full automorphism group.

3. The Picard Group

In this section, we determine the (geometric) Picard group of S. We first show that the hyperplane section of $\bar{S}$ is not divisible in the Picard group. Recall that the canonical class of S is the pull-back of the hyperplane class of $\bar{S}$.

Lemma 5. *The canonical divisor class of S is a primitive vector in* $\operatorname{Pic} S$.

Proof. Let K_S be a canonical divisor of S; since $(K_S)^2 = 16$, it suffices to show that there is no divisor R on S such that $K_S \sim 2R$. We argue by contradiction and suppose that such a divisor R exists. By the Riemann-Roch formula we deduce that

$$h^0(S, \mathcal{O}_S(R)) + h^2(S, \mathcal{O}_S(R)) \geq 6,$$

and from Serre duality it follows that $h^2(S, \mathcal{O}_S(R)) = h^0(S, \mathcal{O}_S(R))$; thus

$$h^0(S, \mathcal{O}_S(R)) \geq 3.$$

Note also that $2 \dim |R| \leq \dim |2R| = 6$, and therefore $h^0(S, \mathcal{O}_S(R)) \in \{3, 4\}$. The image S' of S under the rational map determined by the linear system $|R|$ is an irreducible, non-degenerate subvariety of $|R|^\vee$. Let k be the number of independent quadratic equations vanishing along S'. The image of $\operatorname{Sym}^2 \mathrm{H}^0(S, \mathcal{O}_S(R))$ in $\mathrm{H}^0(S, \mathcal{O}_S(K_S))$ is an $\operatorname{Aut}(S)$-invariant subspace. We easily see that the non-trivial $\operatorname{Aut}(S)$-invariant subspaces of $\mathbb{P}^6$ are $V(a_1, a_2, a_3, c)$ and $V(b_1, b_2, b_3)$. It follows that the image of $\operatorname{Sym}^2 \mathrm{H}^0(S, \mathcal{O}_S(R))$ in $\mathrm{H}^0(S, \mathcal{O}_S(K_S))$ has dimension

$$\binom{h^0(S, \mathcal{O}_S(R)) + 1}{2} - k$$

and that this number must be in $\{0, 3, 4, 7\}$.

Case 1: $h^0(S, \mathcal{O}_S(R)) = 3$, so that $S' \subset \mathbb{P}^2$.
Since S' is irreducible and non-degenerate, we deduce that S' cannot be contained in two independent quadrics, or it would be degenerate. Thus S' must be defined by $k \leq 1$ quadrics, and this is incompatible with the previous constraints.

Case 2: $h^0(S, \mathcal{O}_S(R)) = 4$, so that $S' \subset \mathbb{P}^3$.
Thus $k \geq 3$. Since S' is irreducible and non-degenerate, any quadric vanishing on S' must be irreducible. Hence, any two independent quadrics vanishing along S' intersect in a scheme of pure dimension one and degree four. Since S' is non-degenerate, its degree is at least three. Because there are at least three independent quadrics vanishing on S', this implies $k = 3$, and S' is a twisted cubic curve. But then the map $\operatorname{Sym}^2 \mathrm{H}^0(S, \mathcal{O}_S(R)) \to \mathrm{H}^0(S, \mathcal{O}_S(K_S))$ is surjective, which implies that the image of S under K_S factors through the image under R. So $\bar{S}$ would have to be a curve, contradicting the fact that it is a surface.

Thus the class of the canonical divisor K_S is not the double of a divisor class on S, and the proof is complete. □

Next we prove that $\operatorname{Pic} S$ is a free abelian group of rank 64 and find a set of generators of a subgroup of finite index. We list some curves on S (we let $a_4 := a_1$, to simplify the notation):

(1) the 48 exceptional curves of the resolution $S \to \bar{S}$ (24 defined over $\mathbb{Q}$, 24 defined over $\mathbb{Q}(i)$);
(2) the 32 strict transforms of the conics in the four hyperplanes $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $c = 0$ (24 defined over $\mathbb{Q}$, 8 defined over $\mathbb{Q}(i)$);

(3) the 12 strict transforms of the genus one curves contained in the three hyperplanes $b_1 = 0$, $b_2 = 0$, $b_3 = 0$ (defined over $\mathbb{Q}(i)$);
(4) the 48 strict transforms of the genus one curves contained in the twelve hyperplanes $a_j = \varepsilon a_{j+1}$ or $a_j = \varepsilon ic$, where $j \in \{1, 2, 3\}$ and $\varepsilon \in \{1, -1\}$ (24 defined over $\mathbb{Q}(\sqrt{2})$, 24 defined over $\mathbb{Q}(i, \sqrt{2})$).

These curves are already described in [vL, p. 48]. Denote by $\mathcal{G}$ the set consisting of the above 140 curves.

Proposition 6. *The geometric Picard group of S is a free abelian group of rank 64, and the curves in $\mathcal{G}$ generate a subgroup of finite index.*

Proof. Since $h^1(S, \mathcal{O}_S) = 0$, the Picard group is finitely generated. Moreover, each torsion class in the Picard group determines an unramified connected cyclic covering $\pi\colon \tilde{S} \to S$ that is trivial if and only if the class in the Picard group is trivial. Any such cover induces a similar cover on $\bar{S}$: the inverse image under π of each (-2)-curve in S consists of a disjoint union of (-2)-curves in $\tilde{S}$, which can be contracted to rational double points, thus obtaining an unramified connected cyclic cover of $\bar{S}$. By [Di, Thm. 2.1] we have $\mathrm{H}_1(\bar{S}, \mathbb{Z}) = 0$, and hence the cyclic covering is trivial. We obtain that all torsion classes in $\operatorname{Pic} S$ are trivial, and $\operatorname{Pic} S$ is torsion free.

From the calculation of the Hodge numbers of S, we deduce that the rank of $\operatorname{Pic} S$ is at most 64. Thus, to conclude, it suffices to show that the intersection matrix of the curves in $\mathcal{G}$ has rank 64. By the adjunction formula, we easily see that the self-intersection of the divisor class of a conic or a genus one curve in our list is -4 (in both cases, we mean the strict transform in S of the corresponding curve). The evaluation of the remaining pairwise intersection numbers of the curves above is straightforward, tedious, and preferably done by computer. We check using a computer that the rank of the intersection matrix of the 140 curves in $\mathcal{G}$ is 64, concluding the proof of the proposition. □

Theorem 7. *The Picard group of S is a free abelian group of rank 64, and it is generated by the classes of curves in $\mathcal{G}$.*

The discriminant of the intersection pairing on $\operatorname{Pic} S$ *is* -2^{28}.

Proof. By Proposition 6, the Picard group is free abelian of rank 64, and the lattice L generated by the classes of the curves in $\mathcal{G}$ is of finite index in the Picard group. It remains to show that L is already the full Picard group.

The computation of the intersection matrix shows that the discriminant of L is -2^{28}. Thus the cokernel of the inclusion $L \to \operatorname{Pic} S$ is a finite abelian 2-group. Hence, to prove the equality $L = \operatorname{Pic} S$ it suffices to prove that the natural morphism $L/2L \to \operatorname{Pic} S/2 \operatorname{Pic} S$ is injective; denote by L_2 its kernel. The Galois group $\operatorname{Gal}(\mathbb{Q}(i, \sqrt{2})/\mathbb{Q})$ acts on the lattice $\operatorname{Pic} S$, since the divisors in $\mathcal{G}$ are defined over $\mathbb{Q}(i,\sqrt{2})$, while S is defined over $\mathbb{Q}$. Denote by $\tilde{G}$ the group of automorphisms of $\operatorname{Pic} S$ generated by G and $\operatorname{Gal}(\mathbb{Q}(i,\sqrt{2})/\mathbb{Q})$. The action of $\tilde{G}$ on $\operatorname{Pic} S$ induces an action of $\tilde{G}$ on L, since $\mathcal{G}$ is stable under the action of both G and $\operatorname{Gal}(\mathbb{Q}(i,\sqrt{2})/\mathbb{Q})$, and hence there is an action of $\tilde{G}$ on $L/2L$. The $\mathbb{F}_2$-vector space $L_2 \subset L/2L$ is invariant under $\tilde{G}$ (it is the kernel of a $\tilde{G}$-equivariant homomorphism). Let $\tilde{G}_2 \subset \tilde{G}$ be a Sylow 2-subgroup. Any representation of a 2-group on an $\mathbb{F}_2$-vector space of positive dimension has a non-trivial fixed subspace. In particular, there is a non-trivial $\tilde{G}_2$-invariant subspace in L_2. Using a computer we check that $(L/2L)^{\tilde{G}_2}$ has dimension one, spanned by the reduction modulo two of the canonical class (note that the canonical class is fixed by the action of $\tilde{G}$ on $\operatorname{Pic} S$). We deduce that if $L_2 \neq 0$, then the canonical divisor class is divisible by two in $\operatorname{Pic} S$. By Lemma 5 we know that this is not the case. It follows that $\operatorname{Pic} S$ is generated by the classes of the curves in $\mathcal{G}$. □

Corollary 8. *The intersection pairing on* $\operatorname{Pic} S$ *is even; in particular, there are no curves of odd degree on the surface* S.

Proof. By Theorem 7 the Picard group of S is generated by the elements of $\mathcal{G}$; since all the elements of $\mathcal{G}$ have even self-intersection, we deduce that the pairing on $\operatorname{Pic} S$ is even. By the adjunction formula, the degree of any curve on S has the same parity as its self-intersection; since the pairing on $\operatorname{Pic} S$ is even, we conclude that every curve on S has even degree. □

As a consequence, also the surface $\bar{S}$ contains no curves of odd degree; van Luijk had already shown that there are no lines on $\bar{S}$, see [vL, Prop. 3.4.11].

Using the explicitly known structure of $\operatorname{Pic} S$ as a Galois module, we see the following.

Theorem 9. *The algebraic part of the Brauer group of* S *is the isomorphic image of the Brauer group of* $\mathbb{Q}$ *in the Brauer group of* S.

Proof. It is well-known that the cokernel of the inclusion of the Brauer group of $\mathbb{Q}$ in the algebraic part of the Brauer group of S is isomorphic to the Galois cohomology group $\mathrm{H}^1(\mathbb{Q}, \operatorname{Pic} S)$. Since $\operatorname{Pic} S$ is torsion free and we found a set of generators of the Picard group of S defined over $\mathbb{Q}(i,\sqrt{2})$, we have

$$\mathrm{H}^1(\mathbb{Q}, \operatorname{Pic} S) = \mathrm{H}^1(\operatorname{Gal}(\mathbb{Q}(i,\sqrt{2})/\mathbb{Q}), \operatorname{Pic} S).$$

A computation in Magma [M] shows that the latter cohomology group vanishes, establishing the result. □

This means that there is no 'algebraic Brauer-Manin obstruction' against weak approximation on S. It would be interesting to investigate the transcendental quotient of the Brauer group.

There are further applications of our explicit knowledge of the Picard group. We freely identify curves on $\bar{S}$ with their strict transforms on S.

Theorem 10.

(1) *All conics on $\bar{S}$ are contained in $\mathcal{G}$.*
(2) *The surface $\bar{S}$ does not contain smooth rational curves of degree 4.*
(3) *All curves of degree 4 and arithmetic genus 1 on $\bar{S}$ are in $\mathcal{G}$. In particular, all such curves are smooth and hence of geometric genus 1.*

Equivalently, $\mathcal{G}$ contains all the integral curves C on S such that $C \cdot K_S \leq 4$.

Proof. Any conic C in $\bar{S}$ must be smooth, since $\bar{S}$ does not contain curves of odd degree. We have $C \cdot K_S = 2$ and $C^2 = -4$ by the adjunction formula. We can enumerate all lattice points in the Picard lattice satisfying these two conditions; this results in 2048 elements. If C is a curve, then it has to have nonnegative intersection with all the curves in $\mathcal{G}$ (except possibly itself). Testing this condition leaves only the 32 known conics in $\mathcal{G}$. The other two statements are proved in the same way.

To see the last statement, observe that for an integral curve C on S, we must have $C \cdot K_S \in \{0, 2, 4, \dots\}$. So if $C \cdot K_S \leq 4$, we have $C \cdot K_S \in \{0, 2, 4\}$. If the intersection number vanishes, then C is an exceptional curve and so $C \in \mathcal{G}$. If $C \cdot K_S = 2$, then the image of C on $\bar{S}$ is a conic, and so $C \in \mathcal{G}$ by statement (1). Otherwise, $C \cdot K_S = 4$, and the image of C on $\bar{S}$ is a curve of degree 4, which has arithmetic genus either zero (then C is a smooth rational curve) or one. These two cases are taken care of by statements (2) and (3), respectively. □

The large rank of the Picard lattice unfortunately makes it impossible to extend the computations in this way to higher-degree curves. In any case, the partial results above suggest the following question.

Question 11. *Are all the curves of genus at most 1 on S contained in $\mathcal{G}$?*

Note that surfaces of general type are conjectured to have only finitely many curves of genus at most 1. In general, this is only known in very few cases, for instance, when Bogomolov's inequality $(K_S)^2 > c_2(S)$ holds (see [Bo]); for surfaces contained in an abelian variety (see [Fa]); for very general surfaces of large degree in projective space (see work of Demailly, Siu, Diverio-Merker-Rousseau, Bérczi on the Green-Griffiths Conjecture). None of these cases covers the surface S: Bogomolov's inequality fails for S since $(K_S)^2 = 16 < 80 = c_2(S)$; the surface S is simply connected and hence is not contained in an abelian variety; the surface S is not a surface in $\mathbb{P}^3$, since the only such surfaces of general type with primitive canonical divisor are the surfaces of degree five, and a plane section of a quintic is a curve of odd degree.

According to the Bombieri-Lang conjecture, the rational points on a variety of general type are not Zariski-dense. This means that all but finitely many rational points on a surface of general type lie on a finite set of curves of genus zero or one on the surface. If the question above has a positive answer, then the only such curves defined over $\mathbb{Q}$ on $\bar{S}$ are the conics corresponding to degenerate cuboids. So the Bombieri-Lang conjecture would then imply that there are only finitely many distinct rational boxes (up to scaling).

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