On codes and designs

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ENCODE Workshop
June 22, 2026
Eindhoven University of Technology, Netherlands

Outline

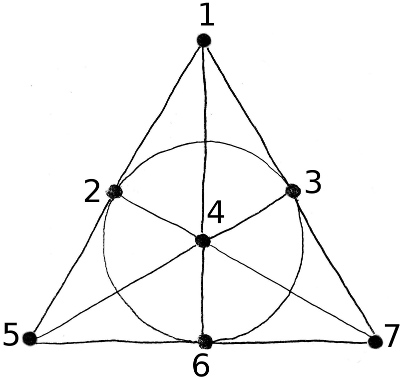
Introduction

Designs in regular semilattices

A new 3-$(42, 6, 1)$ Steiner system

- Let V be a finite set of size v.
- $\binom{V}{k}$ denotes the set of all k-element subsets of V.
  (Notation motivated by $\#\binom{V}{k} = \binom{v}{k}$.)
- Definition.
  $D \subseteq \binom{V}{k}$ is called a t-(v, k, λ) (block) design
  if each $T \in \binom{V}{t}$ lies in exactly λ elements of D.
- elements of D: blocks.
- t: strength of D.
- λ: index of D.
- In the important (and hard) special case $\lambda = 1$:
  D is called a Steiner system.

Example (Fano plane)



$$V = \{1, 2, 3, 4, 5, 6, 7\}$$
$$D = \{\{1, 2, 5\}, \{1, 4, 6\}, \{1, 3, 7\}, \{2, 3, 6\}, \{2, 4, 7\}, \{3, 4, 5\}, \{5, 6, 7\}\}$$

Fano plane D is a 2-$(7, 3, 1)$ design.

Connections between codes and designs

- Assmus–Mattson theorem.
  Let C be a linear code.
  Under certain conditions (essentially: large minimum distance and few dual weights),
  the codewords of fixed weight in C form a t-design.
  In particular: The two extended Golay codes carry 5-designs, including the small and the large Witt designs (which are Steiner systems).
- The blocks of certain designs can be used as parity-check equations for majority-logic decoding.
- Further connections $\rightsquigarrow$ this talk.

Association schemes

Definition of (symmetric) association scheme

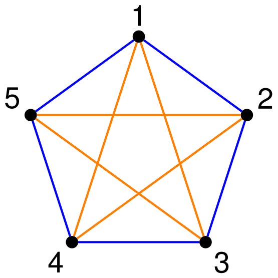
- Ω finite non-empty set.
- $\mathcal{R} = (R_0, \ldots, R_n)$ partition of $\Omega \times \Omega$.

  Think of each R_i as a
  - relation on Ω,
  - graph on vertex set Ω.
- $\mathcal{R}$ is (symmetric) *n*-class association scheme on Ω if
  - (R1) R_0 is the equality relation.
  - (R2) Each R_i is symmetric.
  - (R3) Regularity. For all $i, j, k \in \{0, \ldots, n\}$,

    $$p_{ij}^k := \#\{z \in \Omega \mid (x, z) \in R_i \text{ and } (z, y) \in R_j\}$$

    is constant over all $(x, y) \in R_k$.

Example (Pentagon scheme)



$$\longleftrightarrow \begin{pmatrix} 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} \Omega &= \{1,2,3,4,5\} \\ R_0 &= \{(1,1),(2,2),(3,3),(4,4),(5,5)\} \\ R_1 &= \{(1,2),(2,3),(3,4),(4,5),(5,1), \\ &\qquad (2,1),(3,2),(4,3),(5,4),(1,5)\} \\ R_2 &= \{(1,3),(2,4),(3,5),(4,1),(5,2), \\ &\qquad (3,1),(4,2),(5,3),(1,5),(2,5)\} \end{aligned}$$

$p^2_{12} = 1 \qquad p^1_{11} = 0$ etc.

Standard schemes in combinatorics (1)

Johnson scheme $J(v, n)$

- V set of size $v \geq 2n$
- $\Omega = \binom{V}{n}$
- $x \; R_i \; y \iff \#(x \cap y) = n - i$.

Natural ambient scheme for block designs.

Hamming scheme $H(m, n)$

- Alphabet M of size m
- $\Omega = M^n$
- $\boldsymbol{x} \; R_i \; \boldsymbol{y} \iff d_{\mathrm{Ham}}(\boldsymbol{x}, \boldsymbol{y}) = i$

Natural ambient scheme for block codes.

Standard schemes in combinatorics (2)

Idea of q-analogs in combinatorics

Replace subsets by $\mathbb{F}_q$-subspaces!

q-Johnson (Grassmann) scheme $J_q(v, n)$

q-analog of Johnson scheme $J(v, n)$ $\rightsquigarrow$

- V an $\mathbb{F}_q$-vector space of dimension $v \geq 2n$
- $\Omega = \begin{bmatrix} V \\ n \end{bmatrix}_q$ ($\mathbb{F}_q$-subspaces of dimension n)
- $x\ R_i\ y \iff \dim(x \cap y) = n - i$

Natural ambient scheme for subspace designs.

q-Hamming (bilinear forms) scheme $H_q(m, n)$

- $n \leq m$
- $\Omega = \mathbb{F}_q^{m \times n}$
- $x\ R_i\ y \iff \mathrm{rk}(x - y) = i$

Natural ambient scheme for rank-distance codes.
How is this a q-analog of the Hamming scheme??

Standard schemes in combinatorics (3)

Hamming scheme $H(m, n)$ (reformulated)

- Before: $\Omega = M^n$.
- Equivalent: Let N be a set of size n.
  $\Omega =$ set of maps $N \to M$.

q-Hamming scheme $H_q(m, n)$ (reformulated)

- Before: $\mathbb{F}_q^{m \times n}$.
- Equivalent: $\Omega =$ set of linear maps $\mathbb{F}_q^n \to \mathbb{F}_q^m$.

Bose–Mesner algebra

- Let $D_i \in \mathbb{C}^{\Omega \times \Omega}$ be adjacency matrix of R_i.
- $\mathcal{D} := \{D_0, \ldots, D_n\}$.
- $\mathcal{A} := \langle \mathcal{D} \rangle_{\mathbb{C}}$ Bose–Mesner algebra.
- Properties:
  - closed under multiplication $\implies$ indeed an algebra.
  - commutative.
- $\implies$ set $\mathcal{E} = \{E_0, \ldots, E_n\}$ of primitive idempotents is a $\mathbb{C}$-basis of $\mathcal{A}$.
- Henceforth: Assume fixed order

$$\mathcal{D} = (D_0, \ldots, D_n) \qquad \text{and} \qquad \mathcal{E} = (E_0, \ldots, E_n),$$

typically coming from a metric.

Eigenmatrices

- Two distinguished bases:
  - primal basis $\mathcal{D}$ (adjacency mat., combinatorial side of $\mathcal{A}$)
  - dual basis $\mathcal{E}$ (prim. idempotents, algebraic side of $\mathcal{A}$)
- Change of basis $\mathcal{D} \to \mathcal{E} \rightsquigarrow$ coordinate change matrix P.
- Change of basis $\mathcal{E} \to \mathcal{D} \rightsquigarrow$ coordinate change matrix $\frac{1}{\#\Omega}Q$.
- P and Q (first and second) eigenmatrices of $\mathcal{A}$.
- On transformation level (opposite to coordinate change):

combinatorial side $\xrightarrow{\frac{1}{\#\Omega}Q}$ algebraic side

combinatorial side $\xleftarrow{P}$ algebraic side

coordinates: $\mathcal{D}$ — coordinates: $\mathcal{E}$

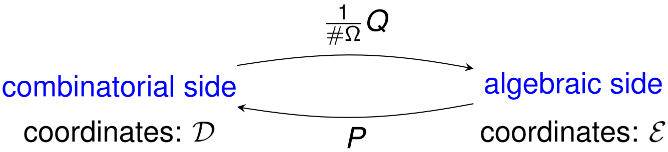
Example (Pentagon scheme)

$$D_0 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} \quad D_1 = \begin{pmatrix} 0&1&0&0&1\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&1&0&1\\ 1&0&0&1&0 \end{pmatrix} \quad D_2 = \begin{pmatrix} 0&0&1&1&0\\ 0&0&0&1&1\\ 1&0&0&0&1\\ 1&1&0&0&0\\ 0&1&1&0&0 \end{pmatrix}$$

$$E_0 = \frac{1}{5}\begin{pmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{pmatrix} \quad E_1 = \frac{1}{5}\begin{pmatrix} 2&\alpha&\beta&\beta&\alpha\\ \alpha&2&\alpha&\beta&\beta\\ \beta&\alpha&2&\alpha&\beta\\ \beta&\beta&\alpha&2&\alpha\\ \alpha&\beta&\beta&\alpha&2 \end{pmatrix} \quad E_2 = \frac{1}{5}\begin{pmatrix} 2&\beta&\alpha&\alpha&\beta\\ \beta&2&\beta&\alpha&\alpha\\ \alpha&\beta&2&\beta&\alpha\\ \alpha&\alpha&\beta&2&\beta\\ \beta&\alpha&\alpha&\beta&2 \end{pmatrix}$$

where $\varphi = \frac{\sqrt{5}+1}{2}$ is the golden ratio and $\alpha := \varphi^{-1}$, $\beta := -\varphi$.

$$\rightsquigarrow \quad \text{eigenmatrices} \qquad P = Q = \begin{pmatrix} 1&2&2\\ 1&\alpha&\beta\\ 1&\beta&\alpha \end{pmatrix}.$$



Association schemes in combinatorics

- Spirit: Interplay of the two sides.
- Source of deep combinatorial results.
- Through involvement of algebraic side:
  Often without transparent combinatorial explanation.

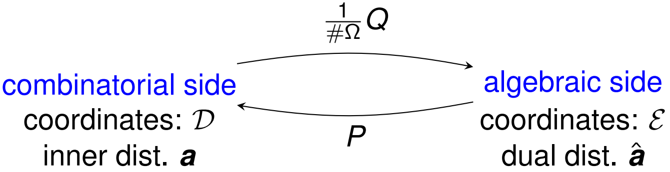
Delsarte theory (1)

- Let $Y \subseteq \Omega$ be non-empty.
- Inner distribution of Y: $\boldsymbol{a} = (a_0, \ldots, a_n)$ where

$$a_i = \frac{1}{\#Y} \cdot \#\{(x, y) \in Y \times Y \mid x \, R_i \, y\}$$

  is average number of $y \in Y$ in relation i to a fixed $x \in Y$.
- Inner distribution lives at the combinatorial side.
- Transform to algebraic side $\rightsquigarrow$ dual distribution of Y:

$$\hat{\boldsymbol{a}} = (\hat{a}_0, \ldots, \hat{a}_n) := Q\boldsymbol{a}.$$



- Delsarte 1973: $\hat{a}_i \geq 0$ (innocent but powerful!)

Delsarte theory (2)

- $C \subseteq \Omega$ is a d-code in $\mathcal{R}$ if $a_1 = a_2 = \ldots = a_{d-1} = 0$.
- Clearly, d-codes in Hamming scheme $H(m, n)$ are m-ary block codes with minimum distance $\geq d$.
- $\rightsquigarrow$ Delsarte's linear programming bound for codes.
  Let $C \subseteq \Omega$ be a non-empty d-code. Then:

$$\#C \leq \text{optimum of the LP}$$

$$\text{maximize} \quad x_0 + \ldots + x_n,$$

with variables $x_0, \ldots, x_n \in \mathbb{R}$, subject to

$$\begin{aligned} x_0 &= 1, \\ x_i &= 0 \quad \text{for all } i \in \{1, \ldots, d-1\}, \\ x_i &\geq 0 \quad \text{for all } i \in \{d, \ldots, n\}, \end{aligned}$$

$$\hat{a}_k \geq 0 \quad \rightsquigarrow \quad \sum_{i=0}^{n} Q_{ki}\, x_i \geq 0 \quad \text{for all } k \in \{0, \ldots, n\}.$$

Delsarte theory (3)

- $D \subseteq \Omega$ is a *t*-design in $\mathcal{R}$ if $\hat{a}_1 = \hat{a}_2 = \ldots = \hat{a}_t = 0$.
- Delsarte 1973. t-designs in the Johnson scheme are the same as combinatorial t-designs. (algebraic "miracle" ...)
- $\rightsquigarrow$ Delsarte's linear programming bound for designs. Let $D \subseteq \Omega$ be a non-empty t-design. Then:

$$\#D \geq \text{optimum of the LP}$$

$$\text{minimize} \quad x_0 + \ldots + x_n,$$

with variables $x_0, \ldots, x_n \in \mathbb{R}$, subject to

$$\hat{a}_k = 0 \quad \rightsquigarrow \quad \sum_{i=0}^{n} Q_{ki}\, x_i = 0 \quad \text{for all } k \in \{1, \ldots, t\},$$

$$x_0 = 1,$$

$$x_i \geq 0 \quad \text{for all } i \in \{1, \ldots, n\},$$

$$\hat{a}_k \geq 0 \quad \rightsquigarrow \quad \sum_{i=0}^{n} Q_{ki}\, x_i \geq 0 \quad \text{for all } k \in \{t+1, \ldots, n\}.$$

Remark on LP-bound

- Hamming and Johnson schemes: Can be very strong.
- But: Harder to evaluate; combinatorially less transparent.
- For q-Johnson, q-Hamming: typically weak.

Remark on Steiner systems

- Steiner systems (i. e., designs of index $\lambda = 1$) coincide with certain classes of extremal codes.
- Johnson scheme: perfect constant-weight codes
- q-Johnson: perfect constant-dimension subspace code
- Hamming scheme: MDS codes
- q-Hamming scheme: MRD codes

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Designs in regular semilattices

A new 3-$(42, 6, 1)$ Steiner system

Designs in regular semilattices

- Joint work with Lukas Klawuhn

Motivation

- Observation.
  Inner distribution of a block design
  = averaged block intersection distribution
  (Mendelsohn, early 1970s).
- Associated Mendelsohn equations:
  Meaning in association scheme framework?
- Follow Delsarte 1976:
  Designs in general $\wedge$-semilattices. Idea:
  - Association scheme does not "know" about intersections of elements.
  - $\rightsquigarrow$ Add $\wedge$-semilattice $(X, \leq)$ beneath Ω.

Setting

- Let $(X, \leq)$ be a non-empty finite graded $\wedge$-semilattice.
- Minimum element $\bot$.
- Height function h.
- Levels $X_i := \{x \in X \mid h(x) = i\}$.
- Top level $\Omega := X_n$ plays a distinguished role.
- Assume the following regularity properties for all r, s:
- $\theta(r, s) := \#\{x \in X_s \mid a \leq x\}$ is constant over all $a \in X_r$.
  Mnemonic: **top**
- $\nu(r, s) := \#\{y \in X_r \mid y \leq b\}$ is constant over all $b \in X_s$.
  Mnemonic: **nether**
- $\mu(r, s) := \#\{x \in X_s \mid a \leq x \leq b\}$
  is constant over all $(a, b) \in X_r \times \Omega$ with $a \leq b$.
  Mnemonic: **middle**

Theorem (Delsarte 1976).

Assume in addition that $(X, \leq)$ is π-regular:

For all $r, s, t \in \{0, \ldots, m\}$ for which there exists a pair $(a, b) \in (X_s \times \Omega)$ with $a \wedge b \in X_r$,

$$\pi(r, s, t) := \#\{(x, y) \in X_t \times \Omega \mid a \leq y \text{ and } x \leq y \text{ and } x \leq b\}$$

is constant over all such pairs $(a, b) \in (X_s \times \Omega)$.

Then the non-empty relations

$$x\ R_i\ y \iff h(x \wedge y) = n - i$$

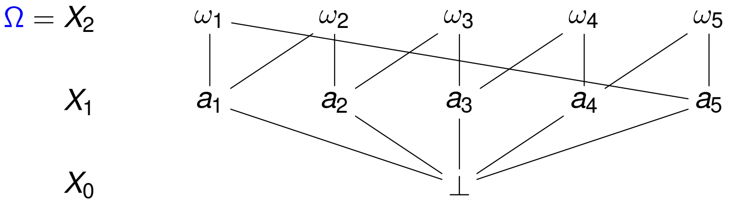
form an association scheme on Ω.

Remark

In this talk, we do not assume π-regularity.
Hence, in general, there is no association scheme on Ω.

Example (Pentagon scheme.)

- The pentagon scheme arises as the top level Ω of the following $\wedge$-semilattice.



- Note.
  - This $\wedge$-semilattice is not π-regular.
  - But Ω is still an association scheme!

Example

All four classical association schemes fit into this framework, i. e. arise as the top level Ω of a suitable $\wedge$-semilattice. (with $x\ R_i\ y \iff h(x \wedge y) = n - i$.)

- Johnson $J(v, n)$
  subset lattice $(V, \subseteq)$, truncated at size n.
- q-Johnson $J_q(v, n)$
  q-analog of the above $\rightsquigarrow$
  subspace lattice $(V, \subseteq)$, truncated at dimension n.
- Hamming $H(m, n)$
  poset of partial maps $N \to M$.
- q-Hamming: $H_q(m, n)$
  q-analog of the above $\rightsquigarrow$
  poset of partial linear maps $\mathbb{F}_q^n \to \mathbb{F}_q^m$.

- **Definition.** $D \subseteq \Omega$ is a t-design of index λ if

$$\#\{B \in D \mid x \leq B\} = \lambda \qquad \text{for all } x \in X_t$$

- **Alternative characterization.**
  For $s \leq t$, define the inclusion matrix W^{st} on the index set $X_s \times X_t$ by

$$W^{st}_{xy} := \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

  Then: D is a t-design $\iff$ $W^{tm}\chi_D = \lambda\mathbf{1}$.
- **Lemma (Suda 2012).** Let D be a t-design of index λ.
  Then for all $r \in \{0, \dots, t\}$, D is an r-design of index

$$\lambda_r := \frac{\theta(r,m)}{\theta(t,m)} \cdot \lambda.$$

- **Definition.** For $a \in X$, define the intersection number
$$\alpha_r(a) := \#\{x \in D \mid h(a \wedge x) = r\}.$$
- **Theorem (K., Klawuhn 2026).**
Mendelsohn equations. For all $r \in \{0, \dots, t\}$,
$$\sum_{s=r}^{n} \nu(r,s)\alpha_s(a) = \nu(r, h(a))\lambda_r.$$
Köhler parametrization. For all $r \in \{0, \dots, t\}$,
$$\alpha_r(a) = \sum_{s=r}^{t} \nu'(r,s)\nu(s, h(a))\lambda_s - \sum_{u=t+1}^{n} \left(\sum_{s=r}^{t} \nu'(r,s)\nu(s,u) \right) \alpha_u(a),$$
where $(\nu) = (\nu(r,s))_{r,s}$ (triangular mat. with unit diagonal) and $\nu'(r,s)$ entries of $(\nu)^{-1}$.
- **Proof (Idea).** Double counting; Gauss elimination.

Theorem (K., Klawuhn 2026).
Let $D \subseteq \Omega$ be a t-Steiner system. Then, for each block $B \in D$, the block intersection distribution at B is given by

$$\alpha_r(B) = \begin{cases} \sum_{s=r}^{t-1} \nu'(r,s)\nu(s,n)\left(\frac{\theta(s,n)}{\theta(t,n)} - 1\right) & \text{if } 0 \le r < t, \\ 0 & \text{if } t \le r < n, \\ 1 & \text{if } r = n. \end{cases}$$

In particular, $\alpha_r(B)$ is constant over all blocks B.

Proof (idea).

- t-Steiner system property forces $\alpha_t(B) = \alpha_{t+1}(B) = \ldots = \alpha_{n-1}(B) = 0$ and $\alpha_n(B) = 1$.
- Plug these values into the Köhler parametrization.

□

Application

The four classical schemes give the distance distributions of

- MDS codes (Forney 1965)
- MRD codes
  - Delsarte 1978, averaged version.
  - Ravagnani 2016/2018, via duality and MacWilliams.
- perfect constant-weight codes (Mendelsohn 1970)
- perfect constant-dimension subspace codes (likely new)

Evaluation of our approach

- Strongest variant of the statements:
  - For individual codewords, no averaging.
  - For unrestricted codes: not only linear or additive ones.
- Unified setting and proof.
- Identifies the natural underlying double-counting objects.

  (Eigenvalue-based approaches
  typically involve more involved expressions.)

Theorem (K., Klawuhn 2026).

Let $D \subseteq \Omega$ and $(\bar{\alpha}_0, \ldots, \bar{\alpha}_n)$ averaged intersection distribution:

$$\bar{\alpha}_r := \frac{1}{\#D} \sum_{x \in D} \alpha_r(x).$$

(a) For all $t \in \{0, \ldots, n\}$,

$$\sum_{s=t}^{n} \nu(t,s)\bar{\alpha}_s \geq \nu(t,n)\frac{\theta(t,n)}{\theta(0,n)} \cdot \#D.$$

(b) D is a t-design $\iff$ the t-th inequality in (a) is sharp.

Proof (Idea).

- Part (a). Apply Cauchy–Schwarz to $\langle W^{tn}\chi_D, \mathbf{1}\rangle$.
- Part (b). Equality criterion of Cauchy–Schwarz.
  + alternative characterization of t-design.

Theorem (K., Klawuhn 2026), repeated.

Let $D \subseteq \Omega$ and $(\bar{\alpha}_0, \ldots, \bar{\alpha}_n)$ its avg. intersection distribution.

(a) For all $t \in \{0, \ldots, n\}$,

$$\sum_{s=t}^{n} \nu(t,s)\bar{\alpha}_s \geq \nu(t,n)\frac{\theta(t,n)}{\theta(0,n)} \cdot \#D.$$

(b) D is a t-design $\iff$ the t-th inequality in (a) is sharp.

Remarks

- Essentially: average intersection distribution
  = Delsarte's inner distribution.
- Advantages of Part (b)
  over Delsarte's t-design characterization $\hat{a}_1 = \ldots = \hat{a}_t = 0$:
  - Single equation on the inner distribution (instead of t).
  - Works even in settings without an association scheme.
- Clearly, $\bar{\alpha}_i \geq 0$.
  $\implies$ Part (a) yields a bound of linear programming type!

Theorem (K., Klawuhn 2026).

Let $D \subseteq \Omega$ be a non-empty t-design. Then:

$$\#D \geq \text{optimum of the LP}$$

$$\text{minimize} \quad x_0 + \ldots + x_n,$$

with variables $x_0, \ldots, x_n \in \mathbb{R}$, subject to

$$\begin{aligned}
\sum_{i=0}^{n} \nu(r, e_i) x_i &= \nu(r, n) \frac{\theta(r, n)}{\theta(0, n)} (x_0 + \ldots + x_n) && (0 \leq r \leq t), \\
x_0 &= 1, \\
x_i &\geq 0 && (0 \leq i \leq n), \\
\sum_{i=0}^{n} \nu(r, e_i) x_i &\geq \nu(r, n) \frac{\theta(r, n)}{\theta(0, n)} (x_0 + \ldots + x_n) && (t < r \leq n)
\end{aligned}$$

Comparison of the LP-bounds

- We can show: Delsarte's LP bound implies our LP bound.
  - Hence: Delsarte's LP bound is potentially stronger.
  - To be investigated:
    Concrete examples where the bounds really differ?
- But: Our bound works even when
  - there is no association scheme on Ω, or
  - the eigenvalues of the association scheme
    are unknown or too complicated.

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A new 3-$(42, 6, 1)$ Steiner system

- Joint work with
  Vedran Krčadinac and Alfred Wassermann.
- Small open case for Steiner systems:

$$3\text{-}(42, 6, 1)$$

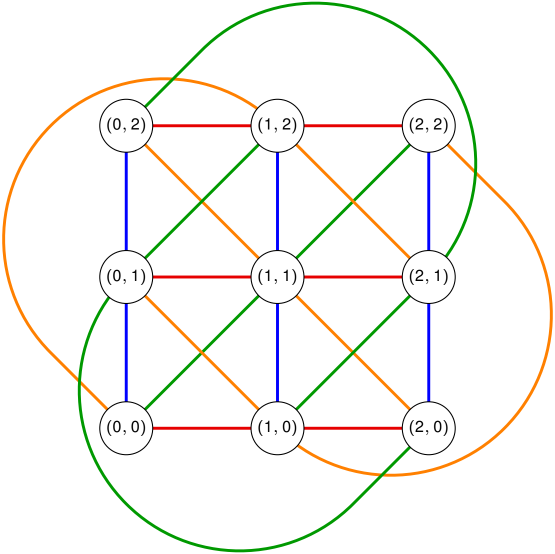
- K., Krčadinac, Wassermann 2025 by computer:

  It exists!
- It is a perfect $(42, 574, 8; 6)$ constant-weight code,
  i. e. constant weight 6, length 42, mindist 8, size 574.
- Now: Analyze computer result.
- Potential reward:
  - structural insight,
  - better: computer-free construction,
  - even better: Infinite series of constructions.

Analysis

- Automorphism group G has order 432, isomorphic to affine linear group $\mathrm{AGL}(2,3)$.
- Suggests description based on $\mathrm{AG}(2,3)$ (affine plane of order 3, Hesse configuration).
- $\mathrm{AG}(2,3)$ has
  - 9 points,
  - 12 lines.

affine plane $\mathrm{AG}(2, 3)$ aka Hesse configuration



- G has two orbits on V:
  - orbit V_A: size 18,
  - orbit V_B: size 24.
- Hence: from given group action,
  - identify the transitive action of $\mathrm{AGL}(2,3)$ on 18 elements,
  - identify the transitive action of $\mathrm{AGL}(2,3)$ on 24 elements,
  - understand the link of those actions.

Orbit V_A

- $18 = 2 \cdot 9$ suggests signed points.
- $\rightsquigarrow V_A = \mathbb{F}_3^\times \times \mathbb{F}_3 \times \mathbb{F}_3$.
- For $\boldsymbol{x} = (x_0, x_1, x_2) \in V_A$:
  - sign $\epsilon = x_0 \in \{\pm 1\}$,
  - point $P = \epsilon \cdot (x_1, x_2) \in \mathbb{F}_3^2$
  - notation P^ϵ.
- Realize $G \cong \mathrm{AGL}(2,3)$ as set of all matrices

$$\det(A) \cdot \begin{pmatrix} 1 & 0 \\ \boldsymbol{b} & A \end{pmatrix} \in \mathrm{SL}(3,3)$$

  with $A \in \mathrm{GL}(2,3)$ and $\boldsymbol{b} \in \mathbb{F}_3^2$.
- Correct action of G on V_A: left multiplication.

Orbit V_B

- $24 = 2 \cdot 12 \rightsquigarrow$ signed lines?
  Fails.
- $24 = 4 \cdot 3! \rightsquigarrow$ ordered parallel classes of lines?
  Fails.
- Frustration.
- A few months later: parallel classes of triangles?
  Works!!

Triangle classes

- Triangles Δ_1, Δ_2 parallel
  $\iff$ their sides are pairwise parallel.
- Parallelism is equivalence relation
  $\rightsquigarrow$ parallel classes of triangles (triangle classes).
- Special feature of $\mathbb{F}_3$:
  Each triangle class has size 3
  and partitions the point set.
- Example: $\begin{pmatrix} A & B & A \\ B & B & C \\ A & C & C \end{pmatrix}$
- Number of triangles $= \frac{9 \cdot 8 \cdot 6}{3!} = 72$.
  $\implies$ number of triangle classes $= \frac{72}{3} = 24$ ✓

Action on V_B

- Model affine plane $\mathrm{AG}(2,3)$ as V_A modulo sign. ($\rightsquigarrow$ standard embedding of $\mathrm{AG}(2,3)$ in $\mathrm{PG}(2,3)$.)
- Let $V_B =$ set of triangle classes in $\mathrm{AG}(2,3)$.
- Induced action of G on V_B is correct!
- The actions on V_A and V_B are linked correctly!

Conclusion

We have identified the geometric ambient setting!

Reverse-engineering the computer result

- Blocks are 6-subsets of $V = V_A \cup V_B$.
- The 574 blocks split into 7 orbits.
- Goal: Geometric descriptions of orbit representatives.
- four “easy” orbits.
- three “harder” orbits, depend on orientation:
  $\mathrm{SL}(2,3)$-action splits set of triangles
  into “clockwise” and “counter-clockwise” ones.

Conclusion

- Geometric construction of a 3-$(42, 6, 1)$ Steiner system.
- Natural $\mathrm{AGL}(2, 3)$-invariance.
- Proof of Steiner system property:
  - Show: all orbits of G on $\binom{V}{3}$ are covered.
  - "Exactly once" then follows from $\#D = 574$.
  - In principle possible by hand, by tedious analysis of 49 orbit representatives.
- Construction remains a miracle.
- Relies on sporadic properties of $\mathrm{AG}(2, 3)$, $\rightsquigarrow$ no extension to an infinite series in sight.
- Remark. Orientation-flip yields a second, isomorphic 3-$(42, 6, 1)$ design invariant under the given $\mathrm{AGL}(2, 3)$-action.

  Computer search: these are the only two.

Thank you!

Slides will be uploaded at
https://mathe2.uni-bayreuth.de/michaelk/