The degree of functions in the Johnson and q-Johnson schemes

Michael Kiermaier

Mathematisches Institut
Universität Bayreuth

DMV Meeting 2023
MS 2 – Combinatorial aspects of finite fields
September 25, 2023
Technische Universität Ilmenau, Germany

joint work with Jonathan Mannaert and Alfred Wassermann

Cameron-Liebler line classes

- Cameron, Liebler 1982:
  "Special" set $\mathcal{L}$ of lines in $\mathrm{PG}(3, q)$.
- Defined by the following equivalent properties:
  - Algebraic property:
    $\chi_{\mathcal{L}} \in \mathbb{R}$-row space of the line-point incidence matrix.
  - Geometric property:
    Constant intersection with any line spread of $\mathrm{PG}(3, q)$.

Various directions of generalization

- Ambient space $\mathrm{PG}(n, q)$.
- lines $\longrightarrow$ k-spaces.
- Allow $q = 1$ (set case).
- points $\longrightarrow$ spaces of degree d.

Goal

Coherent theory of all generalizations.

Subset and subspace lattices

- Fix $q = 1$ (set case) or prime power $q \geq 2$ (q-analog case).
- Fix n non-negative integer.
- Let V be a $\begin{cases} \text{set of size } n \\ \mathbb{F}_q\text{-vector space of dimension } n \end{cases}$
- Let $\mathcal{L}(V)$ be the lattice of all $\begin{cases} \text{subsets of } V \\ \mathbb{F}_q\text{-subspaces of } V \end{cases}$
- For $U \in \mathcal{L}(V)$ let $\mathrm{rk}(U) = \begin{cases} \#U \\ \dim(U) \end{cases}$
- Let $\begin{bmatrix} V \\ k \end{bmatrix} = \{U \in \mathcal{L}(V) \mid \mathrm{rk}(U) = k\}$.
  Set case: $\#\begin{bmatrix} V \\ k \end{bmatrix} = \binom{n}{k} = \begin{bmatrix} n \\ k \end{bmatrix}_1$ Binomial coefficient.
  q-analog case: $\#\begin{bmatrix} V \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q$ Gaussian coefficient.

Association Schemes

- Let X finite set, $\mathcal{R} = \{R_0, \ldots, R_d\}$ partition of $X \times X$.
- $(X, \mathcal{R})$ association scheme if
  - R_0 identity relation
  - All relations R_i are symmetric
  - There exist constants (called intersection numbers) p_{ij}^{ℓ} such that for all $x, y \in X$ with $(x, y) \in R_\ell$

$$\#\{z \in X \mid (x,z) \in R_i \text{ and } (z,y) \in R_j\} = p_{ij}^{\ell}$$

- By definition: Set of adjacency matrices $B^{(i)}$ of R_i pairwise commutable
  $\implies$ are simultaneously diagonalizable
  $\implies$ $\mathbb{R}^X = V_0 \perp \ldots \perp V_d$ orthogonal sum of maximal common eigenspaces

Johnson and Grassmann scheme

- Let $k \leq \frac{n}{2}$ and $X = \begin{bmatrix} V \\ k \end{bmatrix}$.
- For $i \in \{0, \ldots, k\}$ define the relation
  $U_1 \; R_i \; U_2 \iff \mathrm{rk}(U_1 \cap U_2) = k - i$.
- Then $(X, (R_0, \ldots, R_k))$ is a k-class association scheme.

  Set case: Johnson scheme

  q-analog case: Grassmann scheme or q-Johnson scheme.
- Maximal common eigenspaces V_i can be ordered s.t.

$$\bar{V}_i := V_0 \perp \ldots \perp V_i = \mathbb{R}\text{-row space of } W^{(ki)},$$

  where $W^{(ki)}$ is $\begin{bmatrix} V \\ k \end{bmatrix}$-vs-$\begin{bmatrix} V \\ i \end{bmatrix}$ incidence matrix.

The Degree

- Let $f : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{R}$.
- Definition (via algebraic property):
  Degree $\deg(f)$:= smallest d such that $f \in \bar{V}_d$.
- Let $\boldsymbol{x}_U$ be characteristic function of
  $\{K \in \begin{bmatrix} V \\ k \end{bmatrix} \mid U \leq K\}$ ($\mathrm{rk}(U)$-pencil)
- Dually: Let $\bar{\boldsymbol{x}}_U$ be characteristic function of
  $\{K \in \begin{bmatrix} V \\ k \end{bmatrix} \mid U \geq K\}$ (dual $\mathrm{rk}(U)$-pencil).
- Alternative characterization of degree:
  $\deg(f)$ is smallest d
  such that f is a linear combination of d-pencils.

  The (unique) coefficients are called weights $\mathrm{wt}_f(D)$ of f:

$$f = \sum_{D \in \begin{bmatrix} V \\ d \end{bmatrix}} \mathrm{wt}_f(D) \boldsymbol{x}_D$$

- $\rightsquigarrow \deg(f) = 0 \iff f$ constant.

Lemma

- $\deg(\lambda f) = \deg(f)$ *for all* $\lambda \in \mathbb{R} \setminus \{0\}$.
- $\deg(f + g) \leq \max(\deg(f), \deg(g))$
- $\deg(fg) \leq \deg(f) + \deg(g)$

Theorem

Let $\operatorname{rk} I \leq k$ *and* $n - \operatorname{rk} J \leq k$.

- $\deg(\boldsymbol{x}_I) = \operatorname{rk} I$
- $\deg(\bar{\boldsymbol{x}}_J) = n - \operatorname{rk} J$

Proof.

First part: Use that the $\operatorname{Aut}(\mathcal{L}(V))$-orbit of $\boldsymbol{x}_U$ spans $V_{\operatorname{rk} U}$.

Second part:

- Set up linear equation system for the weights, assuming that $\operatorname{wt}(I)$ only depends on $\operatorname{rk}(I \cap J)$.
- Equation system matrix is an invertible triangular matrix.

□

What are the weights of $\bar{\boldsymbol{x}}_U$?

Theorem

Let $i \in \{0, \ldots, k\}$, $J \in \begin{bmatrix} V \\ n-i \end{bmatrix}$, $I \in \begin{bmatrix} V \\ i \end{bmatrix}$ and $z = \mathrm{rk}(I \cap J)$. Then

$$\mathrm{wt}_{\bar{\boldsymbol{x}}_J}(I) = \begin{cases} \delta_{z,k} & \text{if } i = k, \\ (-1)^{i-z} \dfrac{1}{q^{(k-i)(i-z)+\binom{i-z}{2}}} \dfrac{\begin{bmatrix} k-i \\ 1 \end{bmatrix}}{\begin{bmatrix} k-z \\ 1 \end{bmatrix}} \dfrac{1}{\begin{bmatrix} k \\ z \end{bmatrix}} & \text{otherwise.} \end{cases}$$

Proof.

Compute the solutions of the above equation system.
Use negation formula and q-Vandermonde formula for Gaussian coefficients. □

Boolean functions

- Identify sets $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ with their characteristic function $\chi_{\mathcal{F}}$, commonly called Boolean function in this context.
- In this way: Define $\deg(\mathcal{F}) = \deg(\chi_{\mathcal{F}})$.
- Is there a geometric characterization of $\deg(\mathcal{F})$? Suitable generalization of “spread”?

Definition: Design

A set $\mathcal{D} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ is called a t-$(n, k, \lambda)_q$ design,
if every $T \in \begin{bmatrix} V \\ t \end{bmatrix}$ is contained in exactly λ elements of $\mathcal{D}$.

Fact (Delsarte)

$\mathcal{D}$ is a t-$(n, k, \lambda)_q$-design if and only if
$\chi_{\mathcal{D}} \in V_0 \perp V_{t+1} \perp V_{t+2} \perp \ldots \perp V_k$.

Combined with Delsarte's concept of pairwise orthogonality, this leads to:

Fact (Geometric property of the degree)

Let $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$. *If* $d = \deg \mathcal{F}$, *then for each* d-$(n, k, \lambda)_q$ *design* $\mathcal{D}$,

$$\#(\mathcal{F} \cap \mathcal{D}) = \frac{\#\mathcal{F} \cdot \#\mathcal{D}}{\begin{bmatrix} n \\ k \end{bmatrix}}$$

Remark

- Important open question: Is the reverse implication true?
- Would follow if the characteristic functions of d-designs span $V_0 + V_{d+1} + V_{d+2} + \ldots + V_k$.
  (Richness statement about existence of designs)
- Hard question: This would imply Hartman's conjecture from 1987.

Boolean degree 1 functions

- Set case: (Filmus, Ihringer 2019)

  Only the trivial examples $\boldsymbol{x}_P$ and $\bar{\boldsymbol{x}}_H$ ($P \in \begin{bmatrix} V \\ 1 \end{bmatrix}$, $H \in \begin{bmatrix} V \\ n-1 \end{bmatrix}$).
- q-analog case:

  Boolean degree 1 function $=$ Cameron-Liebler set of $(k-1)$-spaces in $\mathrm{PG}(n-1, q)$.

  Non-trivial examples do exist.

  Complete classification probably out of reach.

Change of ambient space

Implication of change of ambient space

- $V \to H \qquad (H \in \begin{bmatrix} V \\ n-1 \end{bmatrix}$ hyperplane$)$
- $V \to V/P \qquad (P \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ point$)$

on the degree?

Theorem

Let $P \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ *and*

$$\mathcal{A} = \{g : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{R} \mid g(K) = 0 \textit{ for all } K \in \begin{bmatrix} V \\ k \end{bmatrix} \textit{ with } P \not\leq K\}.$$

Then

$$\Phi : \mathbb{R}^{\begin{bmatrix} V/P \\ k-1 \end{bmatrix}} \to \mathcal{A}, \qquad \Phi(f) : K \mapsto \begin{cases} f(K/P) & \textit{if } P \subseteq K, \\ 0 & \textit{if } P \not\subseteq K \end{cases}$$

is an isomorphism of $\mathbb{R}$*-vector spaces and*
$\deg_V \Phi(f) = \deg_{V/P}(f) + 1$ *(except certain border cases).*

Proof.

- Everything straightforward, except
  "$\deg_V \Phi(f) \geq \deg_{V/P}(f) + 1$".
- **Lemma.** $P \not\leq D \implies \mathrm{wt}_g(D) = 0$ for all $g \in \mathcal{A}$, $D \in \begin{bmatrix} V \\ \deg(g) \end{bmatrix}$.
- Can be shown using a result of Guo, Li, Wang (2014) stating that the incidence matrices of certain attenuated geometries are of full rank.

Theorem

Let $H \in \begin{bmatrix} V \\ n-1 \end{bmatrix}$ *and*

$$\mathcal{B} = \{g : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{R} \mid g(K) = 0 \text{ for all } g \in \begin{bmatrix} V \\ k \end{bmatrix} \text{ with } K \not\leq H\}.$$

Then

$$\Psi : \mathbb{R}^{\begin{bmatrix} H \\ k \end{bmatrix}} \to \mathcal{B}, \qquad \Psi(f) : K \mapsto \begin{cases} f(K) & \text{if } P \subseteq H, \\ 0 & \text{if } P \not\subseteq H \end{cases}$$

is an isomorphism of $\mathbb{R}$*-vector spaces and*
$\deg_V(\Psi(f)) = \deg_H(f) + 1$ *(except certain border cases).*

Proof.

Follows from the previous theorem by dualization. □

Basic sets of degree d

- Let $I, J \in \mathcal{L}(V)$ with $I \leq J$ and $\operatorname{rk} I + \operatorname{cork} J \leq k$ where corank $\operatorname{cork} J = n - \operatorname{rk} J$.
- Let $\mathcal{F}(I, J) = \{K \in \begin{bmatrix} V \\ k \end{bmatrix} \mid I \leq K \leq J\}$.
- By the above theorems

$$\deg \mathcal{F}(I, J) = \operatorname{rk} I + \operatorname{cork} J.$$

- Basic sets include pencils ($\operatorname{rk} I = 0$) and dual pencils ($\operatorname{cork} J = 0$).

The paired construction

- Construction for the set case $q = 1$.
- Idea: Disjoint union of two "opposite" basic sets.
- Let $I, J \subseteq V$ disjoint, not both empty. Let

$$\mathcal{P}(I,J) = \mathcal{F}(I, J^{\complement}) \uplus \mathcal{F}(J, I^{\complement})$$

- Clear: $\deg \mathcal{P}(I,J) \leq \min(\#I + \#J, k)$.
- There are cases with a strict "$<$"!

Theorem

Let $q = 1$, $I, J \subseteq V$ disjoint, $i = \#I$, $j = \#J$, $k \leq \frac{n}{2}$, $i \leq k \leq n - i$, $j \leq k \leq n - j$.
In the cases

1. $i + j \leq k$ *and* $i + j$ *odd;*
2. $i + j \geq k$ *and* k *odd and* $n = 2k$

we have $\deg \mathcal{P}(I, J) \leq \min(i + j, k) - 1$.

Proof (Idea).

Case 1: Write $\chi_{\mathcal{P}(I,J)}$ as an integer linear combination of basic characteristic functions of degree $i + j - 1$.

Case 2: Induction based on

- $\mathcal{P}(X, Y) = \mathcal{P}(X \uplus \{x\}, Y) \uplus \mathcal{P}(X, Y \uplus \{x\})$ (where $X, Y, \{x\}$ are pairwise disjoint)
- $\mathcal{P}(K, J) = \mathcal{P}(K, \emptyset)$ for $K \in \begin{bmatrix} V \\ k \end{bmatrix}$ and all J.
- Case 1

□

Small sets of degree d

- Natural question:
  Smallest size $m_q(d, k, n)$ of a non-empty set of degree d?
- From $\deg \boldsymbol{x}_D = d$ we get the trivial bound

$$m_q(d, k, n) \leq \begin{bmatrix} n - d \\ k - d \end{bmatrix}.$$

- Trivial bound is sharp for $d = 1$.
- For $q = 1$, $n = 2k$, $d \geq 2$ even, $i = 0$ and $j = d + 1$, the paired construction beats the trivial bound!

Corollary

Let $d \in \{0, \ldots, k - 1\}$ be even. Then

$$m_1(d, k, 2k) \leq 2 \cdot \binom{2k - d - 1}{k}$$

Thank you!

Slides will be uploaded at
https://mathe2.uni-bayreuth.de/michaelk/