Recursive construction of subspace designs

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joint work with Michael Braun, Axel Kohnert and Reinhard Laue

Definition (block design)

Let V be a v-element set.
$D \subseteq \binom{V}{k}$ is a t-(v, k, λ) (block) design
if each $T \in \binom{V}{t}$ is contained in exactly λ elements of D.

q-analog in combinatorics:
Replace subset lattice by subspace lattice!

Definition (subspace design)

Let V be a v-dimensional $\mathbb{F}_q$ vector space.
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Infinite families

Only known infinite nontrivial families with $t \geq 2$:

- $2\text{-}(v, 3, q^2 + q + 1)_q$ for $v \geq 7$, $\gcd(v, 6) = 1$
  (S. Thomas 1987; Suzuki 1990, 1992)
- $2\text{-}(m\ell, 3, q^3 \frac{q^{\ell-5}-1}{q-1})_q$
  for $m \geq 3$, $\ell \geq 7$ and $\ell \equiv 5 \mod 6(q-1)$
  (T. Itoh 1998)

Goal

Construction of new infinite families!

Definition

Fix a parameter set t-$(v, k, \lambda)_q$.

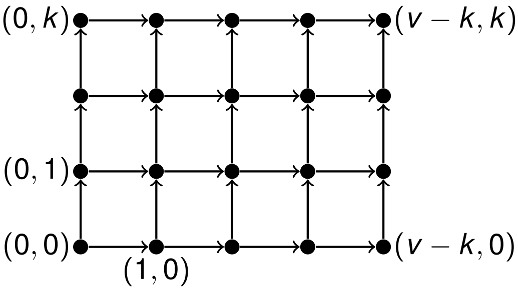
A large set $\mathrm{LS}_q[N](t, k, v)$

is a partition of $\begin{bmatrix} V \\ k \end{bmatrix}_q$ into N t-$(v, k, \lambda)_q$ designs.

Remarks

- $\lambda = \begin{bmatrix} v-t \\ k-t \end{bmatrix}_q / N$ is determined by N, v, k, t, q.
- Only known nontrivial large sets with $t \geq 2$:
  - $\mathrm{LS}_3[2](2, 3, 6)$ (M. Braun 2005)
  - $\mathrm{LS}_2[3](2, 3, 8)$ (M. Braun; A. Kohnert; P. Östergård; A. Wassermann 2014)
  - $\mathrm{LS}_5[2](2, 3, 6)$ (new, computer construction)
- For large sets of *ordinary* block designs:
  Powerful recursion methods!
  (Khosrovshahi, Ajoodani-Namini 1991)
- Adjust those recursion methods to subspace designs!

Definition (Directed grid graph)

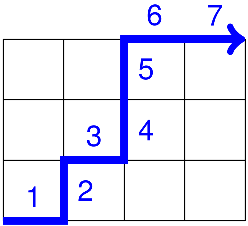


Bijection

$$\text{paths from } (0,0) \text{ to } (v-k,k) \overset{\text{1-to-1}}{\longleftrightarrow} k\text{-subsets } K \text{ of } V$$
$$\text{vertical step} \longleftrightarrow \text{element in } K$$
$$\text{horizontal step} \longleftrightarrow \text{element not in } K$$

Example

$V = \{1, 2, 3, 4, 5, 6, 7\}$.



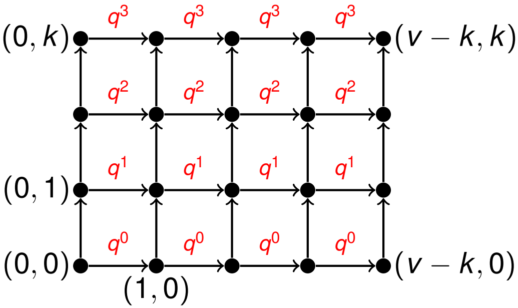
vertical steps: 2, 4, 5 $\rightsquigarrow$ $\{2, 4, 5\} \in \binom{V}{3}$

Question

Is there a q-analog of this bijection?

- Wanted: paths in some graph $\overset{\text{1-to-1}}{\longleftrightarrow}$ $\begin{bmatrix} V \\ k \end{bmatrix}_q$
- As good: paths in some graph $\overset{\text{1-to-1}}{\longleftrightarrow}$ rref in $\mathbb{F}_q^{k\times v}$

Definition (Directed q-grid multigraph)



Bijection

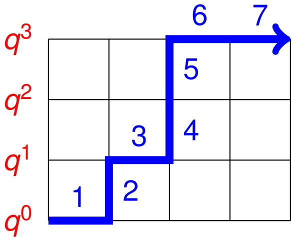
$$k\text{-subspaces } K \text{ of } V \overset{\text{1-to-1}}{\longleftrightarrow} \text{paths from } (0,0) \text{ to } (v-k,k)$$

$$\text{vertical step} \longleftrightarrow \text{pivot column}$$

$$\text{horizontal step} \longleftrightarrow \text{non-pivot column}$$

Example

$V = \mathbb{F}_q^7$



$$\rightsquigarrow \quad \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \end{array}$$

$$\begin{pmatrix} 0 & \mathbf{1} & * & 0 & 0 & * & * \\ 0 & 0 & 0 & \mathbf{1} & 0 & * & * \\ 0 & 0 & 0 & 0 & \mathbf{1} & * & * \end{pmatrix}$$

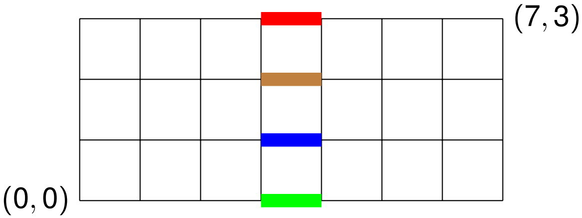
$$\begin{array}{ccccccc} q^0 & & q^1 & & & q^3 & q^3 \end{array}$$

Partitions

Partition of the set of paths from $(0,0)$ to $(v-k,k)$ yields

- partition of $\begin{bmatrix} V \\ k \end{bmatrix}_q$
- identity for Gaussian binomial coefficients
- ... including bijective proof.
- New large sets from old ones!

Example



Partition of paths from $(0,0)$ to $(7,3)$ into 4 parts.

- Blue part $\longleftrightarrow \begin{pmatrix} \multicolumn{5}{c}{1 \times 4 \text{ rref}} & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \multicolumn{6}{c}{\multirow{2}{*}{2×5 rref}} \\ 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$
- Number of such rref: $\begin{bmatrix}4\\1\end{bmatrix}_q \cdot \begin{bmatrix}5\\2\end{bmatrix}_q \cdot q \cdot q^3$
- $\rightsquigarrow$ identity

$$\begin{bmatrix}6\\3\end{bmatrix}_q + q^4 \begin{bmatrix}5\\2\end{bmatrix}_q \begin{bmatrix}4\\1\end{bmatrix}_q + q^8 \begin{bmatrix}4\\1\end{bmatrix}_q \begin{bmatrix}5\\2\end{bmatrix}_q + q^{12} \begin{bmatrix}6\\3\end{bmatrix}_q = \begin{bmatrix}10\\3\end{bmatrix}_q$$

Example

$$\begin{bmatrix}10\\3\end{bmatrix}_q = \begin{bmatrix}3\\0\end{bmatrix}_q \begin{bmatrix}6\\3\end{bmatrix}_q + q^4 \begin{bmatrix}4\\1\end{bmatrix}_q \begin{bmatrix}5\\2\end{bmatrix}_q + q^8 \begin{bmatrix}5\\2\end{bmatrix}_q \begin{bmatrix}4\\1\end{bmatrix}_q + q^{12} \begin{bmatrix}6\\3\end{bmatrix}_q \begin{bmatrix}3\\0\end{bmatrix}_q$$

- Blue part $\longleftrightarrow \begin{pmatrix} \multicolumn{5}{c}{1 \times 4 \text{ rref}} & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \multicolumn{6}{c}{\multirow{2}{*}{2×5 rref}} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
- Describe this set of subspaces as join $\begin{bmatrix}\mathbb{F}_q^4\\1\end{bmatrix}_q * \begin{bmatrix}\mathbb{F}_q^5\\2\end{bmatrix}_q$.
- Simplified notation: $\begin{bmatrix}4\\1\end{bmatrix} * \begin{bmatrix}5\\2\end{bmatrix}$.

$\rightsquigarrow$ Disjoint union of joins

$$\begin{bmatrix}10\\3\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix} * \begin{bmatrix}6\\3\end{bmatrix} \uplus \begin{bmatrix}4\\1\end{bmatrix} * \begin{bmatrix}5\\2\end{bmatrix} \uplus \begin{bmatrix}5\\2\end{bmatrix} * \begin{bmatrix}4\\1\end{bmatrix} \uplus \begin{bmatrix}6\\3\end{bmatrix} * \begin{bmatrix}3\\0\end{bmatrix}$$

Definition

- Let $\mathcal{B}_1, \mathcal{B}_2 \subseteq \begin{bmatrix} V \\ k \end{bmatrix}_q$.
- $\mathcal{B}_1$ and $\mathcal{B}_2$ *t*-equivalent if for all $T \in \begin{bmatrix} V \\ t \end{bmatrix}_q$

$$\#\{B \in \mathcal{B}_1 \mid T \subseteq B\} = \#\{B \in \mathcal{B}_2 \mid T \subseteq B\}$$

Example

$q = 1$, $t = 2$, $k = 3$.

$$\mathcal{B}_1 = \{\{1,2,4\},\{1,3,6\},\{2,5,6\},\{3,4,5\}\}$$
$$\mathcal{B}_2 = \{\{1,2,6\},\{1,3,4\},\{2,4,5\},\{3,5,6\}\}$$

- $T = \{1,4\} \quad \Longrightarrow \quad \lambda(T,\mathcal{B}_1) = \lambda(T,\mathcal{B}_2) = 1$
- $T = \{2,3\} \quad \Longrightarrow \quad \lambda(T,\mathcal{B}_1) = \lambda(T,\mathcal{B}_2) = 0$
- ... Check all T ...
- $\Longrightarrow$ $\mathcal{B}_1$ and $\mathcal{B}_2$ are 2-equivalent.

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Definition

- Let $\mathcal{B} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}_q$
- For $t \geq 0$: $\mathcal{B}$ is (N, t)-partitionable $:\iff$ $\exists$ partition $\{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$ of $\mathcal{B}$ s.t. all $\mathcal{B}_i$ are t-equivalent.
- Always: $\mathcal{B}$ is $(N, -1)$-partitionable.

Lemma

$$\exists \, \mathrm{LS}_q[N](t, k, v) \iff \begin{bmatrix} v \\ k \end{bmatrix} \text{ is } (N, t)\text{-partitionable.}$$

Application for $q \in \{3, 5\}$

- $\exists \, \mathrm{LS}_q[2](2, 3, 6) \implies \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ is $(2, 2)$-partitionable
- *Derived* large set: $\exists \, \mathrm{LS}_q[2](1, 2, 5) \implies \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ is $(2, 1)$-part.
- *Derived* large set: $\exists \, \mathrm{LS}_q[2](0, 1, 4) \implies \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is $(2, 0)$-part.

Rules to create (N, t)-partitionable sets:

Proposition

The disjoint union of (N, t)-partitionable sets
is (N, t)-partitionable.

Basic Lemma

- Let $\mathcal{B}_1$ be (N, t_1)-partitionable
- Let $\mathcal{B}_2$ be (N, t_2)-partitionable

The join $\mathcal{B}_1 * \mathcal{B}_2$ is $(N, t_1 + t_2 + 1)$-partitionable.

Example

$$\begin{bmatrix}10\\3\end{bmatrix} =$$

$$\begin{bmatrix}3\\0\end{bmatrix} * \begin{bmatrix}6\\3\end{bmatrix} \uplus \begin{bmatrix}4\\1\end{bmatrix} * \begin{bmatrix}5\\2\end{bmatrix} \uplus \begin{bmatrix}5\\2\end{bmatrix} * \begin{bmatrix}4\\1\end{bmatrix} \uplus \begin{bmatrix}6\\3\end{bmatrix} * \begin{bmatrix}3\\0\end{bmatrix}$$

$$\underbrace{(2,-1) \quad (2,2)}_{\substack{(2,-1+2+1)\\ =(2,2)}} \quad \underbrace{(2,0) \quad (2,1)}_{\substack{(2,0+1+1)\\ =(2,2)}} \quad \underbrace{(2,1) \quad (2,0)}_{\substack{(2,1+0+1)\\ =(2,2)}} \quad \underbrace{(2,2) \quad (2,-1)}_{\substack{(2,2-1+1)\\ =(2,2)}}$$

$(2,2)$-partitionable

$$\Longrightarrow \begin{bmatrix}10\\3\end{bmatrix} \text{ is } (2,2)\text{-partitionable} \quad \Longrightarrow \exists\, \mathrm{LS}_q[2](2,3,10)$$

Example (cont.)

- $\exists\, \mathrm{LS}_q[2](2,3,10)$ for $q \in \{3,5\}$
- $\implies \exists$ 2-$(10,3,1640)_3$ and 2-$(10,3,48828)_5$ designs
- number of blocks: 238247460880 and 208628946735352

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Theorem

Let $q \in \{3, 5\}$.
There exists an $LS_q[2](2, k, v)$ for all

- $v \equiv 2 \mod 4$ with $v \geq 6$
- $k \equiv 3 \mod 4$ with $3 \leq k \leq v - 3$.

From $LS_2[3](2, 3, 8)$:

Theorem? (work in progress!)

There exists an $LS_2[3](2, k, v)$ for all

- $v \equiv 2 \mod 6$ with $v \geq 8$
- $k \equiv 3, 5 \mod 6$ with $3 \leq k \leq v - 3$.

Open questions

- Construct $\mathrm{LS}_q[2](2,3,6)$, $q \geq 7$ odd.
  (known for $q \in \{3,5\}$, invariant under Singer^2)
- Construct $\mathrm{LS}_2[3](2,4,8)$
  (Smallest open case for $q = 2$, $N = 3$)
- When does $\mathrm{LS}_q[N](1,k,v)$ exist? (includes parallelisms)
  Necessary conditions: $k \mid v$ and $N \mid \begin{bmatrix} v-1 \\ k-1 \end{bmatrix}_q$
  Z. Baranyai 1975: Sufficient for $q = 1$.