Abstract

Short formulas for algebraic curvature tensors via Algebraic Combinatorics

Bernd Fiedler

Eichelbaumstr. 13, D–04249 Leipzig, Germany
Bernd.Fiedler.RoschStr.Leipzig@t-online.de

Algebraic curvature tensors $\mathfrak{R}(x,y,z,u)$ or covariant derivative curvature tensors $\mathfrak{R}'(x,y,z,u,v)$ are covariant tensors of order 4 or 5 which have the same symmetry properties as the Riemann tensor R or its covariant derivative ∇R, respectively.

We show that the spaces of tensors $\mathfrak{R}$ and $\mathfrak{R}'$ are generated by Young symmetrized product tensors

$$\mathfrak{R} :\ y_t^*(U \otimes w)\ ,\ y_t^*(w \otimes U) \quad , \quad \mathfrak{R}' :\ y_{t'}^*(U \otimes W)\ ,\ y_{t'}^*(W \otimes U) \quad (1)$$

where w, W, U are covariant tensors of orders 1, 2, 3. Further W is a symmetric or alternating tensor, whereas U belongs to an irreducible symmetry class characterized by the partition $(2,1)$. y_t and $y_{t'}$ denote the Young symmetrizers of the Young tableaux

$$t \;=\; \begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array} \qquad , \qquad t' \;=\; \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & \end{array} \; .$$

Examples of tensors U are given by

$$\nabla\sigma - \mathrm{sym}(\nabla\sigma) \qquad , \qquad \nabla\tau - \mathrm{d}\tau$$

where σ, τ are symmetric/alternating tensor fields of order 2 and ∇ is a torsion free covariant derivative.

The set of all symmetry classes from which U can be taken forms an infinite 1-parameter family $\mathfrak{S}_\alpha$, $\alpha \in (-\infty, \infty]$. For both $\mathfrak{R}$ and $\mathfrak{R}'$, there exists exactly one value α_0 such that the expressions (1) vanish for all $U \in \mathfrak{S}_{\alpha_0}$. For all other $\alpha \neq \alpha_0$ the tensors $U \in \mathfrak{S}_\alpha$ lead to generators (1) of algebraic tensors $\mathfrak{R}$ and $\mathfrak{R}'$.

The membership $U \in \mathfrak{S}_\alpha$ in a symmetry class $\mathfrak{S}_\alpha$ leads to a set of linear identities

$$\sum_{p \in S_3} c_p(\alpha)\, U_{i_{p(1)} i_{p(2)} i_{p(3)}} \quad = \quad 0 \qquad (2)$$

in which the coefficients $c_p(\alpha)$ are rational functions of α. For $\mathfrak{R}$, $\mathfrak{R}'$ we determine finite sets $\mathcal{A}, \mathcal{A}' \subseteq (-\infty, \infty]$ such that the length of the coordinate representation

$$\mathfrak{R} :\ \sum_{q \in S_4} s_q U_{i_{q(1)} i_{q(2)} i_{q(3)}} w_{i_{q(4)}} \quad , \quad \mathfrak{R}' :\ \sum_{q \in S_5} s_q U_{i_{q(1)} i_{q(2)} i_{q(3)}} W_{i_{q(4)} i_{q(5)}}$$

of an expression (1) can be minimized by means of (2) iff $\alpha \in \mathcal{A}$ or $\alpha \in \mathcal{A}'$. Furthermore, if the length of an expression (1) is minimal then U admits an index commutation symmetry.

Foundation of our investigations is a theorem of S. A. Fulling, R. C. King, B. G. Wybourne and C. J. Cummins which says that the above Young symmetrizers y_t, $y_{t'}$ generate the symmetry classes of algebraic tensors $\mathfrak{R}$, $\mathfrak{R}'$. Furthermore we apply ideals and idempotents in group rings $\mathbb{C}[\mathcal{S}_r]$, the Littlewood-Richardson rule and discrete Fourier transforms for symmetric groups $\mathcal{S}_r$. For certain symbolic calculations we used the Mathematica packages Ricci and PERMS.