COLLECTED PROBLEMS FROM THE PROBLEM SESSION

1. Michael Stoll

Question:
Prove that some explicit prime p does *not* occur as the order of a point in $J(\mathbb{Q})$ as J varies over Jacobian varieties of genus 2 curves over $\mathbb{Q}$.

Remarks:

- Michael Stoll :
  - For elliptic curves we know. Furthermore, all primes $p \leq 29$ occur as order of a point over $J(\mathbb{Q})$.
  - There is a point of order 23 on the Jacobian of the curve
    $$y^2 = 4x^6 + 4x^5 - 7x^4 - 4x^3 + 8x^2 + 8x + 4.$$
    For other primes $p < 29$ one can find curves for a specified prime p in LMFDB such that their Jacobians have a point over $\mathbb{Q}$ of order p.
- Andrew Sutherland :
  - What about curves with real multiplication?
  - What about find a point of order p contained in the image of the Abel-Jacobi map?
  - What about the case when J is geometrically split i.e. $J \simeq E_1 \times E_2$ for some elliptic curves E_1 and E_2?
  - Work of Jef Laga, Ciaran Schembri, Ari Shnidman and John Voight about QM abelian surfaces might be helpful [LSSV23].

2. Nils Bruin

Setup:
Consider the class of all abelian varieties over a fixed base field k of dimension d.

Question:
Is the minimum polarization degree bounded uniformly in d?

Results:
Alexei Skorobogatov : Zarhin's trick implies that the above is a question about the endomorphism ring of $A \times A^\vee$. A theorem by Rémond [Rém17, Theorem 1.1] implies the above using Coleman's conjecture.

The Coleman's conjecture is as follows: Let d and g be positive integers. Consider all abelian varieties A of dimension g defined over number fields of degree d. Then there are only finitely many isomorphism classes among the rings $\mathrm{End}(\bar{A})$, where $\bar{A}$ is the base change of A to the algebraic closure.

3. Andrew Sutherland

Setup:
The aim is to construct a database of abelian surfaces upto isogeny over $\mathbb{Q}$.

Remarks:

- Andrew Sutherland :
  - Assume that the L-function is given and use the property of L function.
  - Jacobians of Genus 2 curves.

  - abelian surfaces arising as a quotient of Jacobians of genus 3 double covers (ramified) of genus 1 curves.
    * $y^2 = f(x^2)$.
    * $y^4 + f(x,z)y^2 + g(x,z) = 0$, where $\deg(f) = 2$ and $\deg(g) = 4$.
- Nils Bruin : abelian surfaces arising as a quotient from an unramified cubic cover of a genus 2 curve.
- Maarten Derickx : Weil restriction over quadratic field of an elliptic curve. Base change of elliptic curve to a cubic extension then take Weil restriction which gives a trace zero variety.
- Tim Dokchitser : Tensor a 2-dim representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ with an elliptic curve $E/\mathbb{Q}$.

4. Héctor H. Pastén Vásquez

Setup:
An $\alpha \in \mathbb{R}$ is called *Diophantine* if there is a variety $X/\mathbb{Q}$ and $f : X \to \mathbb{A}^1$ such that

$$\alpha = \sup_{x \in X(\mathbb{Q})} f(x).$$

Question:

- Does the set D of Diophantine reals form a field?
- Is the set D precisely $\bar{\mathbb{Q}} \cap \mathbb{R}$?

Remarks:
The motivation for the above questions comes from the conjecture that $\mathbb{Z}$ is *not* Diophantine in $\mathbb{Q}$. A conjecture of Mazur implies that the closure of $X(\mathbb{Q})$ in $\mathbb{R}$ is a semi-algebraic set. The difficult part in showing that D is a field is that: if α is Diophantine then so is $-\alpha$.

5. Maarten Derickx

Setup:
Let p be a rational prime and $X_0(p)^+$ be the Atkin-Lehner quotient. Furthermore take p such that $X_0(p)^+$ is not hyperelliptic.

Question: Can one determine the quadratic points on $X_0(p)^+$ for any single value p? For example 97, 109 and 113.

Remarks:

- Nils Bruin : Models of these curves are available. If we know that there are only finitely many quadratic points, then we can look mod p and maybe use Chabauty bounds because of rank.
- Maarten Derickx :
  - Etale fundamental group is abelian so no more extreme form of Chabauty than the *classical one* can be used.
  - $\mathrm{rk}(J_0(97)^+) = 3 =$ genus.
  - $\mathrm{rk}(J_0(109)^+) = 3 =$ genus.

6. Ludwig Fürst

Setup:
Let $g \in \mathbb{N}$, and consider a set with $2g+2$ elements Ω. Define

$$a_g(i) := \#\{\text{partition of } \Omega \text{ into two sets of sizes } i \text{ and } 2g+2-i\}.$$

Then

$$a_g(i) = \begin{cases} \binom{2g+2}{i} & i \neq g+1 \\ \frac{1}{2}\binom{2g+2}{g+1} & i = g+1. \end{cases}$$

Let

$$S_{\pm} := \{a_g(i) \mid i \in \{0, \dots, g+1\},\ i \equiv g \pm 1 \mod 4\}.$$

Then

$$\sum_{n \in S_{\pm}} n = 2^{g-1}(2^g \pm 1).$$

Question:
For any fixed g are $S_{\pm}$ the only sets with sums $2^{g-1}(2^g \pm 1)$?

Remarks:
S_+ and S_- count the even and odd theta characteristics for a curve of genus g respectively. For a generic genus g hyperelliptic curve given by $y^2 = f(x)$ with Theta divisor Θ, the Riemann-Roch space $\mathcal{L}(4\Theta)$ splits as a sum of 1-dimensional eigenspaces corresponding to the Theta characteristics. The numbers $a_g(i)$ are the sizes of the orbits under the action of the Galois group of f. A positive answer to the question would imply that just from the dimensions the (invariant) even/odd subspaces $\mathcal{L}(4\Theta)^{\pm}$ can only split as a sum over the even/odd Theta characteristics. A numerical calculation has shown that the statement holds for $g \leq 90$.

7. Alex Betts

Setup:
Let $X/\mathbb{Q}$ be a "nice" curve and J be the Jacobian variety of X.

Question:
Can one find an example of X as above such that J has no non-zero factor of rank 0 over $\mathbb{Q}$ and

$$X(\mathbb{Q}) \subseteq X(\mathbb{Q}_p)_{\mathrm{CK}} \subseteq X(\mathbb{Q}_p) \quad \text{and} \quad X(\mathbb{Q}_p)_{\mathrm{CK}} \subseteq X(\mathbb{Q})?$$

Remarks:
It would suffice to find an example of a curve $X/\mathbb{Q}$ such that $X(\mathbb{Q}_p)_{\mathrm{Chab}} = X(\mathbb{Q})$ for 100% of primes p. A weaker question is if there is a finite subscheme Z of $X/\mathbb{Q}$ such that $X(\mathbb{Q}_p)_{\mathrm{Chab}} \subseteq Z(\mathbb{Q}_p)$ for 100% of primes p?

Héctor H. Pastén Vásquez : Are there sub-linear bounds on the size of Chabauty set available?

8. Samir Siksek

Setup:
Let Φ_n be the n-th cyclotomic polynomial. Define a *super cyclotomic polynomial* as any polynomial of the form

$$X^a \prod_{i=1}^{m} \Phi_{n_i}(X)^{b_i}$$

for a, m, b_i, n_i non-negative integers.

Question:
Find all *ternary linear identities* between super cyclotomic polynomials.

Remarks:

- An example is $\Phi_2^4\Phi_5 - \Phi_1^4\Phi_{10} = 10X\Phi_4^3$.
- For a prime $p \nmid n$, $\Phi_n(\zeta_p) \in \mathcal{O}(\mathbb{Q}(\zeta_p), S)^\times$ where $S := \{1 - \zeta_p\}$. Therefore substituting in the above linear identity between super cyclotomic polynomials we get $\epsilon + \delta = 1$ where ϵ and δ are S-units. This can be used to construct elliptic curves with very few primes of bad reduction.
- For more identities and further information see [SV23].

9. Tony Várilly-Alvarado

Question:
Does there exist a number field k and quadrics $Q_1, Q_2, Q_3 \in k[x_0, \ldots, x_7]$ such that their smooth complete intersection $Z(Q_1, Q_2, Q_3) \subseteq \mathbb{P}^7$ is $\bar{k}$-irrational?

Remarks:
The answer is yes over the complex numbers, and one expects the answer should be yes for a very general such intersection. The motivation is that one can "hope" to use the above to construct an example of a 4-fold which is geometrically irrational but has positive proportion of rational specialisations over $\bar{\mathbb{F}}_p$. For this one will have to assume *Tate's conjecture* on *cycle class groups* i.e. $\mathrm{Pic}(X) \otimes \mathbb{Q}_l \overset{\simeq}{\to} \mathrm{H}^2_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l)^{G_k}$, where G_k is the absolute Galois group of k.

Isabel Vogt pointed out that work of Colliot-Thelénène and Pirutka on quadric surface bundle fourfolds could be adapted to give a positive answer to this question.

10. Tim Dokchitser

Setup:
A conjecture of Debes-Deschamps states that if $G \to G'$ is a surjective homomorphism of finite groups that is also split then any $K/\mathbb{Q}$ realising G' as a Galois group, i.e. $\mathrm{Gal}(K/\mathbb{Q}) = G'$, embeds into a G-extension of $\mathbb{Q}$.

Question:
Is this true for

$$1 \to Q_8 \to \mathrm{GL}_2(\mathbb{F}_3) \to S_3 \to 1?$$

Remarks:
If the above is false then this provides a counterexample to the above conjecture. All S_3 extensions have been classified and Q_8 extensions are very well understood (see [JLY02]).

11. Tom Fisher

Question:
Let p be a rational prime. Does there exist absolutely simple Jacobian varieties with arbitrary large p-part of the Tate-Shafarevich group?

Remarks:
The question is solved by Victor Flynn and Ari Shnidman in [FS22], if we replace Jacobian varieties with abelian varieties.

12. Alexei Skorobogotav

Setup:
Let $X \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ be a given "nice" variety defined by $t_0 X_0^d + t_1 X_1^d + t_2 X_2^d + t_3 X_3^d = 0$ such that X_p is ELS.

Question:
Can one "somehow" determine if there are enough rational points on X, i.e., it is hard to find points on average for example by showing that the average height of the smallest point is very large?

13. Michael Stoll

Setup:
Let $m \in \mathbb{Z}$ and define the sequence $\mathcal{A}(m) : (a_0(m), \ldots, a_k(m), \ldots)$ given by $a_0(m) = m$ and $a_{k+1}(m) \coloneqq a_k(m)^2 - 1$ for $k \geq 0$. Let

$$T(m) \coloneqq \{p \mid p \text{ is prime and } \exists n \geq 0 \text{ such that } p | a_n(m)\}.$$

Let $T \coloneqq \bigcap_{m \in \mathbb{Z}} T(m)$.

Question:
Is T infinite?

Remarks:
$T \cap \{1, \ldots, 10^5\} = \{2, 3, 7, 23, 19207\}$.

References

[SV23] Samir Siksek and Robin Visser, *Curves with few bad primes over cyclotomic $\mathbb{Z}_\ell$-extensions*, 2023. ↑3

[Rém17] Gaël Rémond, *Conjectures uniformes sur les variétés abéliennes*, The Quarterly Journal of Mathematics **69** (2017), no. 2, 459-486. ↑1

[JLY02] Christian U. Jensen, Arne Ledet, and Noriko Yui, *Generic polynomials*, Mathematical Sciences Research Institute Publications, vol. 45, Cambridge University Press, Cambridge, 2002. Constructive aspects of the inverse Galois problem. ↑4

[FS22] Victor E. Flynn and Ari Shnidman, *Arbitrarily large p-torsion in Tate-Shafarevich groups*, 2022. ↑4

[LSSV23] Jef Laga, Ciaran Schembri, Ari Shnidman, and John Voight, *Rational torsion points on abelian surfaces with quaternionic multiplication*, 2023. ArXiv preprint https://arxiv.org/abs/2308.15193. ↑1