# Collected Problems from Problem Session

Rational Points 2019

July 15, 2019

## 1 David Harari

**Setup** Let $k$ be a number field, $S$ a finite set of places of $k$, $U/k$ smooth geometrically integral and $U \hookrightarrow X$ be a smooth proper compactification. Let $F/k$ be a finite commutative group scheme with Cartier dual $\hat{F}$. Let $[Y] \in \mathrm{H}^1(U, F)$ be an $F$-torsor and $(P_v)_{v\in S} \in \prod_{v\in S} U(k_v)$. For every $v$, define $E_v := \mathrm{im}[U(k_v) \to \mathrm{H}^1(k_v, F), P_v \mapsto [Y](P_v) =: f_v]$, which is *not* a subgroup in general. For $v \notin S$, $E_v \supseteq \mathrm{H}^1(\mathcal{O}_v, F)$ if $S$ is large enough. Let $b \in \mathrm{H}^1(k, \hat{F})$ and $B_Y := \{b \cup [Y]\} + \mathrm{Br}(k) \subset \mathrm{Br}_1(U)$ and $B = B_Y \cap \mathrm{Br}(X)$.

Consider the following conditions:

a) $(P_v)_{v\in S} \in \overline{U(k)} \subseteq \prod_{v\in S} U(k_v)$

b) There exists $a \in \mathrm{H}^1(k, F)$ such that $a_v = f_v$ for all $v \in S$ and $a_v \in E_v$ for $v \notin S$.

c) There exists $a \in \mathrm{H}^1(k, F)$ such that $a_v = f_v$ for all $v \in S$ and $a_v \in \langle E_v \rangle$ for $v \notin S$.

d) There is no BMO for $(P_v)_{v\in S}$ associated to $B$.

One has a) $\Longrightarrow$ b) $\Longrightarrow$ c). By Poitou-Tate, c) $\Longleftrightarrow$ d).

**Question** Does c) $\Longrightarrow$ b)? Note that there is a big difference between $E_v$ and $\langle E_v \rangle$ in general!

**Results**

- "Yes" if $|F|$ is prime. $G$ finite $k$-group, $G^{\mathrm{ab}} := G/D(G) =: F$. $\mathrm{SL}_n \to Y \to \mathrm{SL}_n / G =: U$. $E_v = \mathrm{im}[\mathrm{H}^1(k_v, G) \to \mathrm{H}^1(k_v, F)]$, $B = \mathrm{Br}_1(X)$ [Dem10]
- b)$\not\Rightarrow$a) [DAN17]
- For $F = \mu_n$ + mild conditions: answer should be "yes".

## 2 Felipe Voloch

**Question** Fix a field $K$ and $n \geq 4$. Let $L/K$ be a finite separable field extension of degree $n$, i. e. $\mathrm{Tr}_{L/K} \neq 0$. Let $\alpha \in L^\times$ and $a, b \in \{0, 1\}$. Are there $x, y \in L$ such that $xy = \alpha$ and $\mathrm{Tr}_{L/K}(x) = a, \mathrm{Tr}_{L/K}(y) = b$? (It suffices to consider $a, b \in \{0, 1\}$ by homogeneity of the trace.) For which $L/K$ is the answer "yes" for all $\alpha \in L^\times$?

**Example** Let $a = b = 0$. The system of equations $xy = \alpha, \mathrm{Tr}_{L/K}(x) = 0, \mathrm{Tr}_{L/K}(y) = 0$ defines a projective hypersurface $X_{\alpha,L}$ of degree $n - 1$ in $\mathbf{P}^{n-2}_K$ and the question is equivalent to $X_{\alpha,L}(K) \neq \emptyset$.

**Results** Answer "yes" if $K$ is finite and $n \geq 5$ (J. Sheekey and G. v. d. Voorde).

**Question** How special are such $X_{\alpha,L}$ among all hypersurfaces of degree $n - 1$ in $\mathbf{P}^{n-2}_K$?

## 3 Victor Flynn

**Question** Let $A/\mathbf{Q}(t_1, \ldots, t_r)$ be an (absolutely) simple abelian variety (assume $\dim A > 1$ and $A$ not constant to make the question non-trivial). Is there a $\mathbf{Q}$-specialisation which is (absolutely) simple again?

**Solution** (found by Bjorn Poonen) Can even preserve the geometric endomorphism ring [Mas96]. (The paper [MP12, Proposition 1.13] solves the question with $\mathbf{Q}$ replaced by an algebraically closed field of characteristic 0.)

## 4 Nils Bruin

**Known** [Rut13]

$$\#\{f \in \mathbf{Z}[s,t]_4 : I(f) = 0,\ 0 < |J(f)| < X\}/\operatorname{GL}_2(\mathbf{Z}) = CX + O_\varepsilon(X^{5/6+\varepsilon})$$

**Question** What happens if we replace the condition $0 < |J(f)| < X$ by $J(f) = AB^2C^3$, $0 < |A|, |B|, |C| < X$, $A, B, C$ square free, pairwise coprime? Results with $C$ removed, or results about counting the number of forms by square free part would also be useful.

## 5 Kamal Makdisi

**Question** Let $k$ be a global field (already $k = \mathbf{Q}$ is interesting), and let $\mathscr{A} \in \operatorname{Br}(k)$ be a central simple algebra. There is an analytic proof of

$$\sum_{v \in M_k} \operatorname{inv}_v(\mathscr{A}) = 0 \tag{1}$$

via the $\zeta$-function of $\mathscr{A}$, i.e. that the sequence in Albert-Brauer-Hasse-Noether is a complex (Reference: original article by Hasse; Weil, *Basic Number Theory*; Vigneras, *Arithmetic of Quaternion Algebras*). Let $X/k$ be a nice variety with $\mathscr{A} \in \operatorname{Br}(X)$ such that $X(\mathbf{A}_k)^{\mathscr{A}} = \emptyset$, which proves $X(k) = \emptyset$. Does the analytic proof of (1) suggest an analytic approach to proving $X(k) = \emptyset$ in this situation?

## 6 Christopher Frei

**Known** Lefton [Lef79] has proved the following bound for irreducible cubic polynomials with cyclic Galois group:

$$\{f = ax^3 + bx^2 + cx + d \in \mathbb{Z}[x] \ :\ \text{irreducible},\ \operatorname{Gal}(f/\mathbb{Q}) = A_3,\ |a|, |b|, |c|, |d| \le X\} \ll_\varepsilon X^{3+\varepsilon}.$$

**Problem** Let $\ell \in \mathbb{N}$ and replace the size conditions on $a, b, c, d$ by

$$|a|, |d| \le X^{1/\ell}, \quad |b|, |c| \le X.$$

In this situation, prove the bound $\ll_{\ell,\varepsilon} X^{1+2/\ell+\varepsilon}$.

**Remark** The condition $\operatorname{Gal}(f/\mathbb{Q}) = A_3$ means that the discriminant of $f$ is a square,

$$b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd = z^2.$$

Lefton fixed $a, b, c$, bounded the number of integral points of bounded height on the resulting affine quadratic curve in $d, z$, and then summed it up.

The same approach for the problem above gives $\ll_\varepsilon X^{2+1/\ell+\varepsilon}$, and I also know how to get $\ll_{\varepsilon,\ell} X^{3/2+2/\ell}$. To get the bound $X^{1+2/\ell}$, one could try to fix $a, b, d$ and bound the number of integral points on the resulting cubic curve in $c, z$. The fact that we are averaging over $a, b, d$ might help.

## 7 John Cremona

**Question** Replace $D_f = \square \in \mathbf{Z} \setminus \{0\}$ in the previous problem by $D_f = \square \in \mathbf{R} \setminus \{0\}$. Is there a closed formula for

$$\operatorname{vol}\left\{(a_0, a_2, \ldots, a_{n-1}) \in [-1,1]^n : D_f > 0, \text{where } f = \sum_{i=0}^{n-1} a_i x^i\right\}/2^n?$$

**Results** For $n = 3$: easy exercise involving $\log 2$ (for $f = ax^2 + 2bx + c$: volume $\in \mathbf{Q}$). For $n = 4$: practise. For $n = 5$: should be enough for general even degree.

Known for characteristic polynomial of a "random" $n \times n$-matrix with entries Gaussian distributed. (Reference?)

## 8 Andrew Sutherland

**Setup** Let $K$ be a number field, $A/K$ an (absolutely) simple abelian variety of dimension $g > 1$. For $\mathfrak{p}$ a finite prime of $K$ of good reduction denote by $\overline{A}_{\mathfrak{p}}$ the reduction of $A$ mod $\mathfrak{p}$.

**Question** (local-global question for being a Jacobian) Suppose the isogeny class of $\overline{A}_{\mathfrak{p}}$ contains a Jacobian (or principally polarized abelian variety) for all but finitely many primes $\mathfrak{p}$ of $K$. Does the isogeny class of $A$ contain a Jacobian (or principally polarized abelian variety)?

**Motivation** An affirmative answer to this question would give an effective day/night algorithm that takes as input the $L$-function of $A$ and outputs either a curve with the same $L$-function or a proof that no such curve exists.

**Remarks** Start with the particularly interesting cases $g = 2, 3$, then $\dim \mathscr{M}_g = 3g - 3 = \frac{g(g+1)}{2} = \dim \mathscr{A}_{g,1}$. The larger $g$ is, the harder it becomes for a PPAV to be a Jacobian, and we expect to get a counterexample.

## 9 Daniel Loughran

**Setup** Let $U/\mathbf{Q}$ be a smooth surface with $\mathbf{G}_m^2 \subseteq U$.

**Question** Is the Brauer-Manin obstruction the only one to strong approximation?

**Consequence/application** Erdös-Straus conjecture $\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

**Results** Yes if $U$ is projective [San81]. True if $\mathbf{G}_m^2$-action extends. Probably true if one assumes $\overline{K}[U]^\times = \overline{K}^\times$ and $\mathrm{Pic}(U)$ torsion-free. (later: wrong! Erdös-Straus conjecture gives counterexample to the question)

## References


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