PROBLEM SESSION - RATIONAL POINTS 2017

Let K be a field, a curve C over K is called *nice* if it is smooth projective and geometrically irreducible.

1. Bjorn Poonen

Let $A = \mathbb{F}_2[x,y]/(y^2 + xy + x^3 + x^2 + x)$ and $K = \text{Frac}A$, then $\text{Pic}\, A = \mathbb{Z}/2\mathbb{Z}$. and is generated by (x, y). Let K_∞ be the completion of K and $I \subseteq A$ and define

$$\alpha := \sum_{a \in \alpha, a \neq 0} \frac{1}{a}, \quad \beta_I := \sum_{a \in I, a \neq = 0} \frac{1}{a} \text{ in } K_\infty.$$

If $I = cA$ for some $c \in A$ then $\alpha = c\beta$ and if $I = (x, y)$ then using Drinfeld modules one can proof that $\alpha^2 + x\alpha\beta + x\beta^2 = 0$.

Question: Can one prove this using more elemetry methods?

Question: If so, do these metheds generalize to $A = O_C(C \setminus \{\infty\})$ for C a nice curve over a finite field $\mathbb{F}_q$ and $\infty \in C(\mathbb{F}_q)$.

2. Felipe Voloch

Let C, D nice curves over a finite field $\mathbb{F}_q$ of genus ≥ 2. The goal is to investigate wether one can detect if there exists a non constant morphism $D \to C$ using etale descent. To be precise, let $K = \mathbb{F}_q(D)$ then the non constant morphisms are in bijection with $C(K) \setminus C(\mathbb{F}_q)$. If one furthermore lets $C(\mathbb{A}_K)^{f-cov} \subset \mathbb{C}(\mathbb{A}_K)$ be the locus cut out by etale descent.

Question: does one have: $C(K) \neq C(\mathbb{F}_q) \Leftrightarrow C(\mathbb{A}_K)^{f-cov} \neq C(\mathbb{F}_q)$?

Let $C^{(1)}$ and $D^{(1)}$ denote the set of closed points of the schemes C and D. Then one has a surjection

$$C(\mathbb{A}_K) = \prod_{v \in D^{(1)}} C(K_v) \twoheadrightarrow \prod_{v \in D^{(1)}} C(\mathbb{F}_q(v)).$$

Elements in the right most product give rise to a map $D^{(1)} \to C^{(1)}$. If one now takes an element $f \in C(\mathbb{A}_K)$ such that the associated map $D^{(1)} \to C^{(1)}$ is non-surjective and non-constant. Then it is clear that this f does not come from $C(K)$.

Question: Can one at least show that for $f \in C(\mathbb{A}_K)$ that induce a non-surjective and non-constant map $D^{(1)} \to C^{(1)}$ one has that $f \notin C(\mathbb{A}_K)^{f-cov}$.

3. Andrew Sutheland

Let C_1 and C_2 be the following nice genus 3 curves over $\mathbb{Q}$:

$$C_1 : y^2 + (x^4 + x^3 + x^2 + 1)y = x^7 - 8x^5 - 4x^4 + 18x^3 - 3x^2 - 16x + 8$$
$$C_2 : x^3 z - x^2 y^2 + 2x^2 yz - x^2 z^2 - xy^3 + 2xy^2 z - yz^3 = 0$$

These both have prime discriminant ± 8233 and there are at the moment no known curves of genus 3 with smaller prime discriminant. They happen to have the same Euler factors for all primes $p < 2^{26}$ and computing the period matrices numerically seems to suggest the existence of an isogeny $J(C_1) \to J(C_2)$ whose kernel is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/4\mathbb{Z}$. Their jacobians both have Mordell Weil and analytic rank 0 and their endomorphism rings over $\overline{\mathbb{Q}}$ are both $\mathbb{Z}$.

Question: Is there any connection between these curves other then that they happen to have isogenous Jacobians.

Question: Can one determine $J(C_1)(\mathbb{Q})_{tors}$ and $J(C_2)(\mathbb{Q})_{tors}$? Yes, Michael Stoll computed that they are both isomorphic to $\mathbb{Z}/36\mathbb{Z}$.

Question: Is there a practical algorithm that allows one to efficiently compute $J(C)(\mathbb{Q})_{tors}$ for an arbitrary genus 3 curve?

4. Daniel Loughran

Question: Can one prove the Manin conjecture for the Burkhardt quartic?

To be more precise: The Burkhardt quartic is the hypersurface $X \subseteq \mathbb{P}^4$ given by:

$$f := y_0(y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3) - 3y_1y_2y_3y_4 = 0$$

Let $H = \det\left((\frac{\partial^2 f}{\partial y_i \partial y_j})_{i,j=0}^4 \right)$ be its Hessian, and $U \subseteq X$ the locus where H is nonzero. Define the height function on $h : \mathbb{P}^4(\mathbb{Q}) \to \mathbb{R} \geq 0$ by

$$h(y_0 : y_1 : y_2 : y_3 : y_4) = \prod \max_i |y_i|_v$$

where v runs through all valuations of $\mathbb{Q}$ and for $B \in \mathbb{R}_{\geq 0}$ define

$$N(B) := \{x \in U(\mathbb{Q}) : H(x) \leq B\}.$$

Question: can one prove that there exists some r such that $N(B) \sim B(\log B)^r$

5. David Holmes

Let $(R, \mathfrak{m}_R)$ be a regular $\mathfrak{m}_R$-adically complete local ring. Let $r \in R$ be a non-zero element. Let $A = R[[x, y]]/(xy - r)$. Then A is R-flat and is a complete normal local noetherian domain, as stated in 7.8.3 on page 215 of EGA III part II. Our aim is to classify the principal ideals of A which become trivial after base-change over R to $K := \mathrm{Frac} R$. More precisely, we show:

Theorem 5.1. *Let $a \in A$ be an element such that $a \otimes 1$ is a unit in $A \otimes_R K$. Then there exist*

- *an element $s \in R$;*
- *non-negative integers m, n such that $mn = 0$;*
- *a unit $u \in A^\times$;*

such that $a = sx^n y^m u$.

The proof of this theorem can be found at `https://arxiv.org/abs/1402.0647` The proof also works if 'regular' is replaced by 'unique factorization domain'.

Question: Does it remain true if we assume R is normal, or log regular?

A positive answer has nice applications for constructing separated qoutients of Pic.

6. Bonus question

Can one find the $\mathbb{Q}$ rational points on the curves given by

$$xy^3 - x(x-1)^3y^2 + (x-1)(3x^2-1)y + x^2(x-1) = 0 \text{ and}$$

$$y^2 = x(4x^{12} - 23x^{11} + 58x^{10} - 95x^9 + 82x^8 - 124x^7 + 136x^6 - 17x^5 - 34x^4 - 45x^3 + 30x^2 + 5x - 4)$$

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