Chabauty without the Mordell-Weil group

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Schney
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$$C_p \colon 5y^2 = 4x^p + 1$$

are the obvious ones (namely, ∞ and $(1, \pm 1)$),
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(Dahmen and Siksek show this for $p = 7$ and $p = 19$
and deal with $p = 11$ and $p = 13$ in another way, assuming GRH.)

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We compute its 2-Selmer group $\mathrm{Sel}_2 J_{17} \cong (\mathbb{Z}/2\mathbb{Z})^2$.
Since $J_{17}(\mathbb{Q})[2] = 0$, this gives $\operatorname{rank} J_{17}(\mathbb{Q}) \leq 2$.
We know the point $[(1, 1) - \infty]$ of infinite order, so $\operatorname{rank} J_{17}(\mathbb{Q}) \geq 1$,
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What can we do now?

Idea: Use method of Poonen-Stoll for concrete curves
(but without integration).

Method

More general setting:

$C/\mathbb{Q}$ nice curve with Jacobian J;

$P_0 \in C(\mathbb{Q})$, gives embedding $\iota\colon C \hookrightarrow J$;

$\Gamma \subset J(\mathbb{Q})$ a subgroup with saturation $\overline{\Gamma}$;

p a prime number; $X \subset C(\mathbb{Q}_p)$, e.g., a residue disk.

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p a prime number; $X \subset C(\mathbb{Q}_p)$, e.g., a residue disk.

For $P \in J(\mathbb{Q}_p)$ set

$$q(P) = \{\pi_p(Q) : Q \in J(\mathbb{Q}_p), \exists n \geq 0 \colon p^n Q = P\} \subset \frac{J(\mathbb{Q}_p)}{pJ(\mathbb{Q}_p)}$$

and for $S \subset J(\mathbb{Q}_p)$ set $q(S) = \bigcup_{P \in S} q(P)$.

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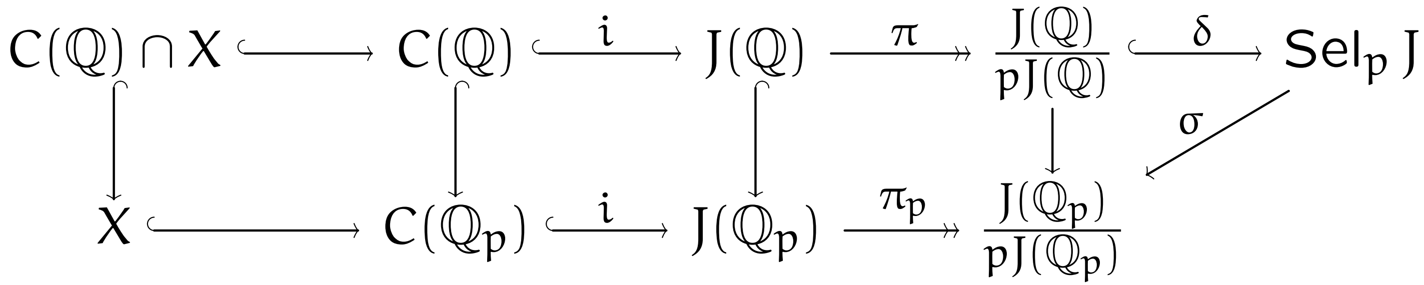

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so $Q_n \in \Gamma + \ker(\sigma\delta\pi) \overset{(1)}{\subset} \Gamma + pJ(\mathbb{Q})$, which leads to T_{n+1} and Q_{n+1}.

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so $Q_n \in \Gamma + \ker(\sigma\delta\pi) \overset{(1)}{\subset} \Gamma + pJ(\mathbb{Q})$, which leads to T_{n+1} and Q_{n+1}.

(ii) $\psi \colon J(\mathbb{Q}) \to J(\mathbb{Q})/\bar{\Gamma}$ free; $\forall n \geq 0$: $\psi(i(P)) = p^n \psi(Q_n)$, so $\psi(i(P)) = 0$. $\square$

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Corollary.
If $P_0 \in X$, X is contained in (half) a residue disk,
$\ker \sigma \subset \delta\pi(J(\mathbb{Q})[p^\infty])$ and $q(X + J(\mathbb{Q})[p^\infty]) \cap \mathrm{im}\,\sigma \subset \pi_p(J(\mathbb{Q})[p^\infty])$, then

$$C(\mathbb{Q}) \cap X = \{P_0\}.$$

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- If C is given as $y^2 = f(x)$ and $L = \mathbb{Q}[x]/\langle f \rangle$, then have compatible maps
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  where $L_2 = L \otimes_{\mathbb{Q}} \mathbb{Q}_2$ and $R^{\square} = R^{\times}/(R^{\times})^2$.
- Can compute $\mathrm{Sel}_2\, C$ and $\mathrm{Sel}_2\, J$ as a subset and subgroup of $L^{\square}$.
- So work with $L^{\square}$ and $L_2^{\square}$ instead of $J(\mathbb{Q})/2J(\mathbb{Q})$ and $J(\mathbb{Q}_2)/2J(\mathbb{Q}_2)$.

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Remark. Can leave out 2-adic condition for $\mathrm{Sel}_2\, J$.

Applications

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(5) More to come!

Thank You!