JACOBS UNIVERSITY

How to Find the Rational Points on a Rank 1 Genus 2 Curve

Michael Stoll
Jacobs University Bremen

Lorentz Center, Leiden, May 14, 2007

The Goal

Let $C/\mathbb{Q}$ be a smooth projective curve of genus 2, given by

$$y^2 = f(x) = f_6 x^6 + f_5 x^5 + f_4 x^4 + f_3 x^3 + f_2 x^2 + f_1 x + f_0 \,.$$

Goal: Determine $C(\mathbb{Q})$!

Assumptions: Let J be the Jacobian of C.

- rank $J(\mathbb{Q}) = 1$, and a generator G of $J(\mathbb{Q})$ (mod torsion) is known;
- We know a point $P_0 \in C(\mathbb{Q})$.

For simplicity, we will assume that $J(\mathbb{Q}) = \mathbb{Z} \cdot G$.

Remark.
If $C(\mathbb{Q})$ is non-empty, then P_0 is usally easy to find.
If $C(\mathbb{Q})$ is empty, there are ways to prove this fact.

The Idea

Let $\iota : C \longrightarrow J, \quad P \longmapsto [P - P_0]$

be the embedding determined by the basepoint P_0.

We have to determine the set

$$R = \{n \in \mathbb{Z} : nG \in \iota(C)\} = \phi\big(C(\mathbb{Q})\big) \subset \mathbb{Z}\,,$$

where $\phi : C(\mathbb{Q}) \xrightarrow{\iota} J(\mathbb{Q}) \xrightarrow{\cong} \mathbb{Z}$.

Outline of Procedure:

1. Find N such that $R \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/N\mathbb{Z}$ is injective;
2. For each coset $k + N\mathbb{Z}$,
   either exhibit a point $P \in C(\mathbb{Q})$ with $\phi(P) \in k + N\mathbb{Z}$,
   or show that $R \cap (k + N\mathbb{Z})$ is empty.

Step 1

We don't know how to do Step 1 in general.

However, we can hope to find a suitable N in our case,
or more generally, when $\operatorname{rank} J(\mathbb{Q}) < g(C)$.

The idea here is to use Chabauty's Method:

Let p be a prime. There is a pairing

$$\Omega^1_J(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p\,, \qquad (\omega, R) \longmapsto \int_0^R \omega\,.$$

Since $\operatorname{rank} J(\mathbb{Q}) = 1$ and $\dim_{\mathbb{Q}_p} \Omega^1_J(\mathbb{Q}_p) = 2$, there is a differential

$$0 \neq \omega_p \in \Omega_C(\mathbb{Q}_p) \cong \Omega^1_J(\mathbb{Q}_p)$$

that kills $J(\mathbb{Q}) \subset J(\mathbb{Q}_p)$.

How to Find N

Theorem.
If the reduction $\bar{\omega}_p$ does not vanish on $C(\mathbb{F}_p)$ and $p > 2$, then each residue class contains at most one rational point.

This implies that $C(\mathbb{Q}) \to J(\mathbb{Q})/NJ(\mathbb{Q})$ is injective, where $N = \big(J(\mathbb{Q}) : J(\mathbb{Q}) \cap J(\mathbb{Q}_p)^1\big)$.

Heuristically, the set of primes p satisfying this condition should have positive density (at least when J is simple):
Note that for a random $\bar{\omega} = \dfrac{(a + bx)\,dx}{y}$, there is a $\approx 50\%$ chance.

Heuristic/Conjecture 1.
If J is simple, then there are primes $p > 2$ such that $\bar{\omega}_p \neq 0$ on $C(\mathbb{F}_p)$.

In practice, this works very well.

How to Compute $\bar{\omega}_p$

Given a prime p of good reduction, we find $\bar{\omega}_p$ as follows.

Let $K \subset \mathbb{P}^3$ be the Kummer Surface of J: $J \xrightarrow{\pi} K = J/\{\pm 1\}$.

Compute the image of NG on K;
it will have the form $\pi(NG) = (p^2 a : p^2 b : p^2 c : d)$ with $p \nmid d$.

We have $ax^2 - bx + c \equiv \lambda(\alpha x + \beta)^2 \bmod p$; and

$$\bar{\omega}_p = \frac{(\bar{\alpha}x + \bar{\beta})\,dx}{y}.$$

Remarks.

1. We can compute $\pi(NG)$ from $\pi(G)$.
2. We can do the computation mod p^3 (i.e., efficiently even for large N).

Step 2

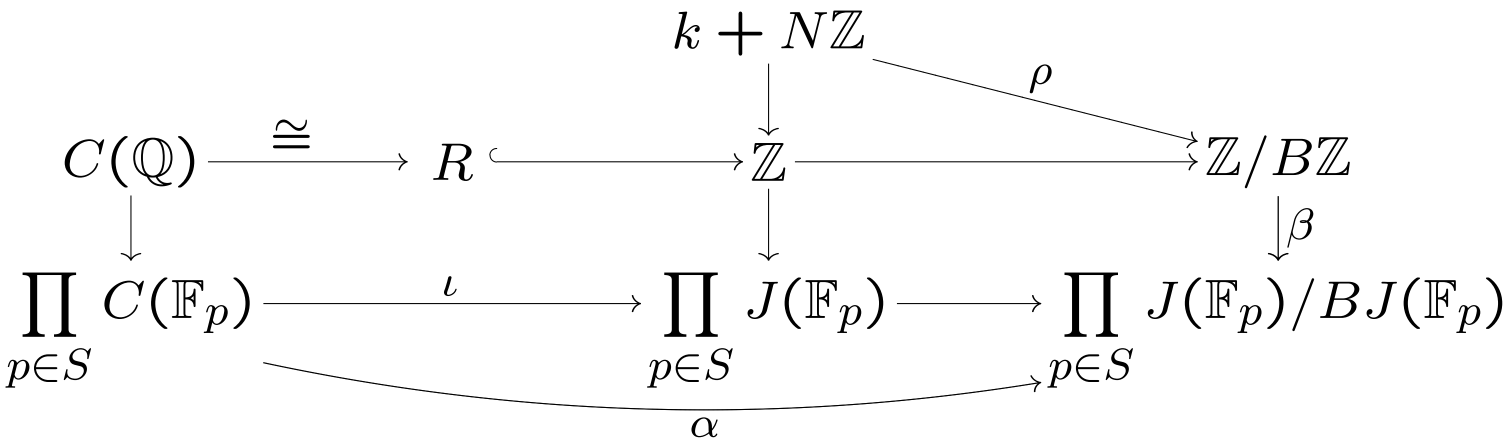
Given a coset $k + N\mathbb{Z}$, we let k_0 be the absolutely smallest representative and check whether $k_0 G \in \iota(C)$.
(Before embarking on a potentially costly exact computation of $k_0 G$, we check for several primes p whether its image mod p is in $\iota\Big(C(\mathbb{F}_p)\Big)$.)

If so, we have found $P_k = \iota^{-1}(k_0 G) \in C(\mathbb{Q})$;
this is then the only rational point in this residue class.

Otherwise, we try to prove that $R \cap (k + N\mathbb{Z}) = \emptyset$
by a Mordell-Weil Sieve computation.

Mordell-Weil Sieve

Let S be a finite set of primes of good reduction.
Let B be a multiple of N. Consider the following diagram.



If the images of $\beta \circ \rho$ and of α do not intersect, then $R \cap (k + N\mathbb{Z}) = \emptyset$.

Heuristic/Conjecture 2:
If $R \cap (k + N\mathbb{Z}) = \emptyset$, then this will be the case
when B and S are sufficiently large.

Practical Remarks

- To avoid combinatorial explosion,
  we compute $\beta^{-1}\big(\mathrm{im}(\alpha)\big)$ successively for a sequence
  $1 = B_0, B_1, \ldots, B_n = B$, where $B_m = q_m B_{m-1}$ with q_m a prime.

- When B_m is a multiple of N,
  we check the smallest point in the class if it comes from C;
  if so, we can discard everything in the same coset mod N.

- We can work with several values of N at the same time.

Conclusion

- Given a curve C of genus 2, a point in $C(\mathbb{Q})$ and a generator of $J(\mathbb{Q})$, there is an algorithm that computes $C(\mathbb{Q})$.

- Termination of the algorithm is conditional on two conjectures; these conjectures are supported by heuristics and experimental evidence.

- In practice, the procedure works and is quite efficient.

  For example, for the "Flynn-Poonen-Schaefer Curve"

  $$C : y^2 = x^6 + 8\,x^5 + 22\,x^4 + 22\,x^3 + 6\,x^2 + 5\,x + 1\,,$$

  it takes about 1.5 seconds to find $\#C(\mathbb{Q}) = 6$.

- Step 2 does not require the "Chabauty Condition" $r < g$. So if we can do Step 1 for a given curve C, we are in good shape to find $C(\mathbb{Q})$.