# Rational Points on Curves of Genus 2: Experiments and Speculations

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# Curves of Genus 2

A curve of genus 2 over $\mathbb{Q}$ is given by an equation

$$C : y^2 = f_6 x^6 + f_5 x^5 + f_4 x^4 + f_3 x^3 + f_2 x^2 + f_1 x + f_0$$

with $f_j \in \mathbb{Z}$, such that $(f_6, f_5) \neq (0, 0)$
and the polynomial on the right does not have multiple roots.

A rational point on this curve $C$
is a pair of rational numbers $(\xi, \eta)$ satisfying the equation.

In addition, there can be rational points "at infinity",
corresponding to the square roots of $f_6$ in $\mathbb{Z}$.

We denote the set of rational points on $C$ by $C(\mathbb{Q})$.

**Theorem** (Mordell's Conjecture, proved by Faltings). $C(\mathbb{Q})$ is finite.

# The Questions

Consider curves of genus 2 over $\mathbb{Q}$:

$$C : y^2 = f_6x^6 + f_5x^5 + f_4x^4 + f_3x^3 + f_2x^2 + f_1x + f_0$$

with $f_j \in \mathbb{Z}$, and of size $\max_j |f_j| \le N$.

**Question.**
What can we say about $C(\mathbb{Q})$, the set of rational points on $C$, as $N$ grows?

- How many rational points are there on average?
- What is the distribution of the number of points?
- What is the largest number of rational points?

- How are the sizes of the points distributed?
- How large can the points get?

# Heuristics (1)

The condition that the point $(\frac{a}{b}, \frac{c}{b^3})$ is on $C$
translates into a linear condition on the coefficients $f_j$:

$$a^6 f_6 + a^5 b f_5 + a^4 b^2 f_4 + a^3 b^3 f_3 + a^2 b^4 f_2 + a b^5 f_1 + b^6 f_0 = c^2$$

The curves satisfying this correspond to points in the intersection of a coset of a 6-dimensional lattice in $\mathbb{R}^7$ with a cube of side length $2N$.

We can estimate the size of this set by the volume of the corresponding slice of the cube, divided by the covolume of the lattice.

We obtain for the average number of points with $x = \frac{a}{b}$:

$$\mathbb{E}_{(a:b)}(N) \sim \frac{\gamma(a:b)}{\sqrt{N}} \qquad \text{as } N \to \infty.$$

with $\gamma(a:b)$ of order $H(a:b)^{-3}$,
where $H(a:b) = \max\{|a|, |b|\}$ is the height of $x$.

# Heuristics (2)

We let

$$\gamma(H) = \sum_{\substack{(a:b)\in\mathbb{P}^1(\mathbb{Q}) \\ H(a:b)\leq H}} \gamma(a:b) = \gamma - O\left(\frac{1}{H}\right)$$

where

$$\gamma = \sum_{(a:b)\in\mathbb{P}^1(\mathbb{Q})} H(a:b) = \lim_{H\to\infty} \gamma(H) \approx 4.79991\,.$$

We obtain for the average number of points with $H(x) \leq H$:

$$\mathbb{E}_{\leq H}(N) \sim \frac{\gamma(H)}{\sqrt{N}} \qquad \text{as } N \to \infty.$$

# A First Conjecture

Naive approach gives

$$\mathbb{E}_{\leq H}(N) = \frac{\gamma(H)}{\sqrt{N}} + o(N^{-1/2}) \qquad \text{for } H \ll N^{6/5-\varepsilon}.$$

Let $\mathbb{E}(N)$ denote the average number of rational points.

**Corollary.** $$\liminf_{N\to\infty} \sqrt{N} \cdot \mathbb{E}(N) \geq \gamma\,.$$

**Conjecture 1.** $$\lim_{N\to\infty} \sqrt{N} \cdot \mathbb{E}(N) = \gamma\,.$$

Can the exponent $6/5$ be improved? How far?

Stephan Baier: Using quadratic Gauss sum, gets $7/5 - \varepsilon$.

# Large Points (1)

If we accept Conjecture 1, then we should expect about

$$(\gamma - \gamma(H))2^7 N^{13/2} = O\Big(H^{-1} N^{13/2}\Big)$$

curves of size $\le N$ that have points of height $\ge H$.

So generically, we expect that rational points on a curve of size $N$ will have height $\ll N^{13/2+\varepsilon}$.

**Conjecture 2.**

Given $\varepsilon > 0$, there is $B_\varepsilon > 0$ and a Zariski-open subset $\emptyset \ne U_\varepsilon \subset \mathbb{A}^7$ such that the rational points on every curve of size $\le N$ whose coefficient vector is in $U_\varepsilon$ have height $\le B_\varepsilon N^{13/2+\varepsilon}$.

# Large Points (2)

It is, however, likely that there are families of curves with larger points.

A naive dimension count predicts a family with points of height $\gg N^9$ (maybe not over $\mathbb{Q}$).

Still, we can hope that the following is true.

**Conjecture 3.**
There are $\kappa > 0$ and $B > 0$ such that every rational point on a genus 2 curve of size $N$ has height $\leq BN^\kappa$.

Andrew Granville:
True under ABC with $\kappa = 1/2$ for quadratic twists of a fixed curve.

Su-Ion Ih: Conjecture 3 follows from Vojta's Conjecture.

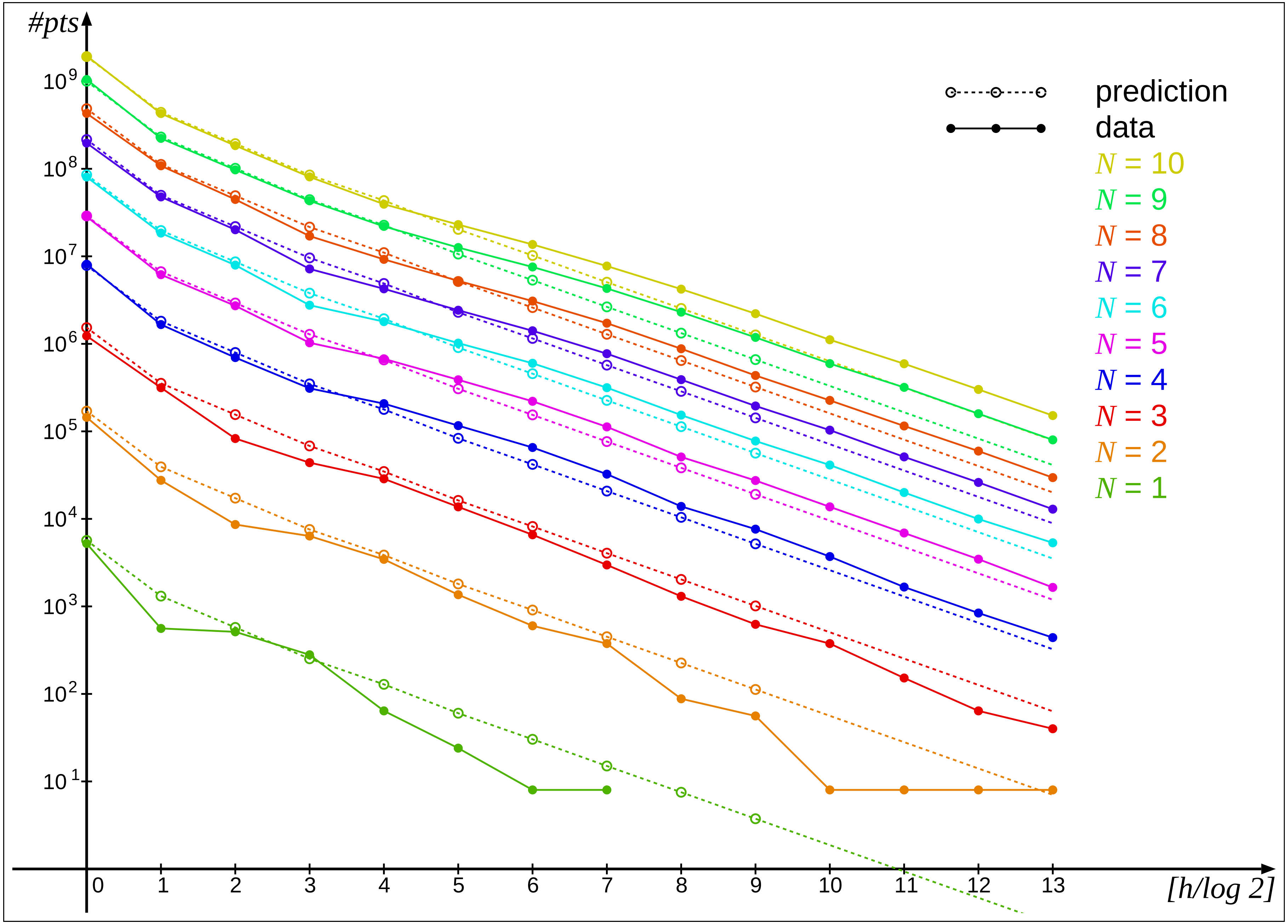
# Data

Nils Bruin and I have found (very likely) all rational points on all genus 2 curves of size $N \leq 3$. Records:

| size of curves | $N = 1$ | $N = 2$ | $N = 3$ |
|---|---|---|---|
| max. $H(P)$ | 145 | 10 711 | 209 040 |
| max. $H(P)/N^{13/2}$ | 145.00 | 118.34 | 165.55 |

Using an efficient implementation of point search (ratpoints), I have found all rational points of height $< 16384$ on all genus 2 curves of size $N \leq 10$.

The graph on the next slide compares the counts for points in the height brackets $2^n \leq H < 2^{n+1}$ with the heuristic prediction.


#pts
$10^9$
$10^8$
$10^7$
$10^6$
$10^5$
$10^4$
$10^3$
$10^2$
$10^1$
0
1
2
3
4
5
6
7
8
9
10
11
12
13
[h/log 2]
prediction
data
$N$ = 10
$N$ = 9
$N$ = 8
$N$ = 7
$N$ = 6
$N$ = 5
$N$ = 4
$N$ = 3
$N$ = 2
$N$ = 1

# Observations

- Overall good agreement with heuristic prediction.
- "Too many" relatively large points.

The deviation might be related to the existence of families with "overly large" points.

Note that agreement is good in the range $H \leq N^{3/2}$.

# Number of Points

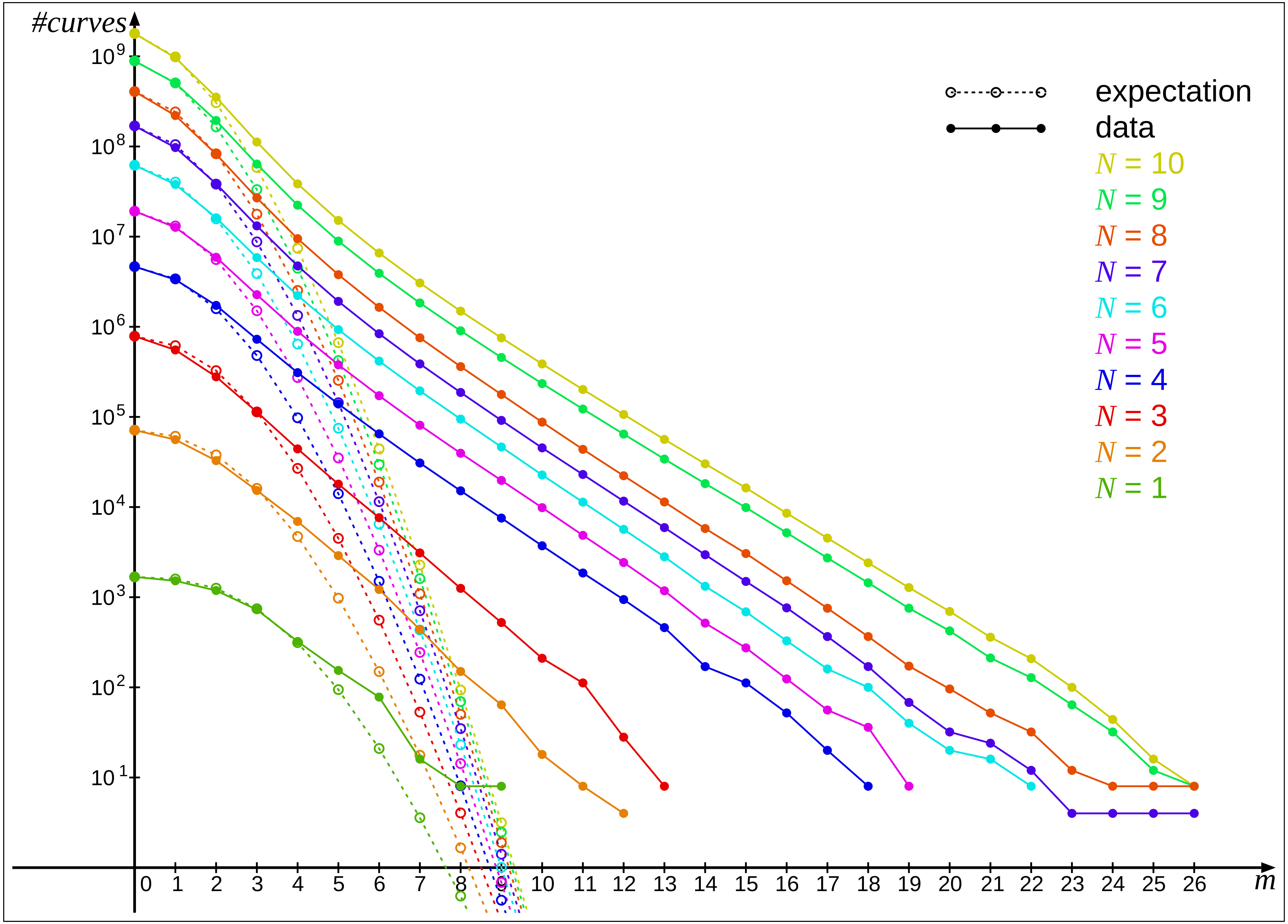
The following graph shows the number of curves of size $\leq N$
with at least a given number of point pairs.

Under the assumption that the events

"there is a rational point with $x$-coordinate $x_0$"

are independent for all $x_0$, one arrives at a prediction for these numbers.

It turns out that the assumption must be wrong.



#curves
$10^9$
$10^8$
$10^7$
$10^6$
$10^5$
$10^4$
$10^3$
$10^2$
$10^1$
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26
m
expectation
data
N = 10
N = 9
N = 8
N = 7
N = 6
N = 5
N = 4
N = 3
N = 2
N = 1

# Another Conjecture

It appears that there is a fairly constant probability that a curve with at least $m$ point pairs actually has at least $m + 1$.

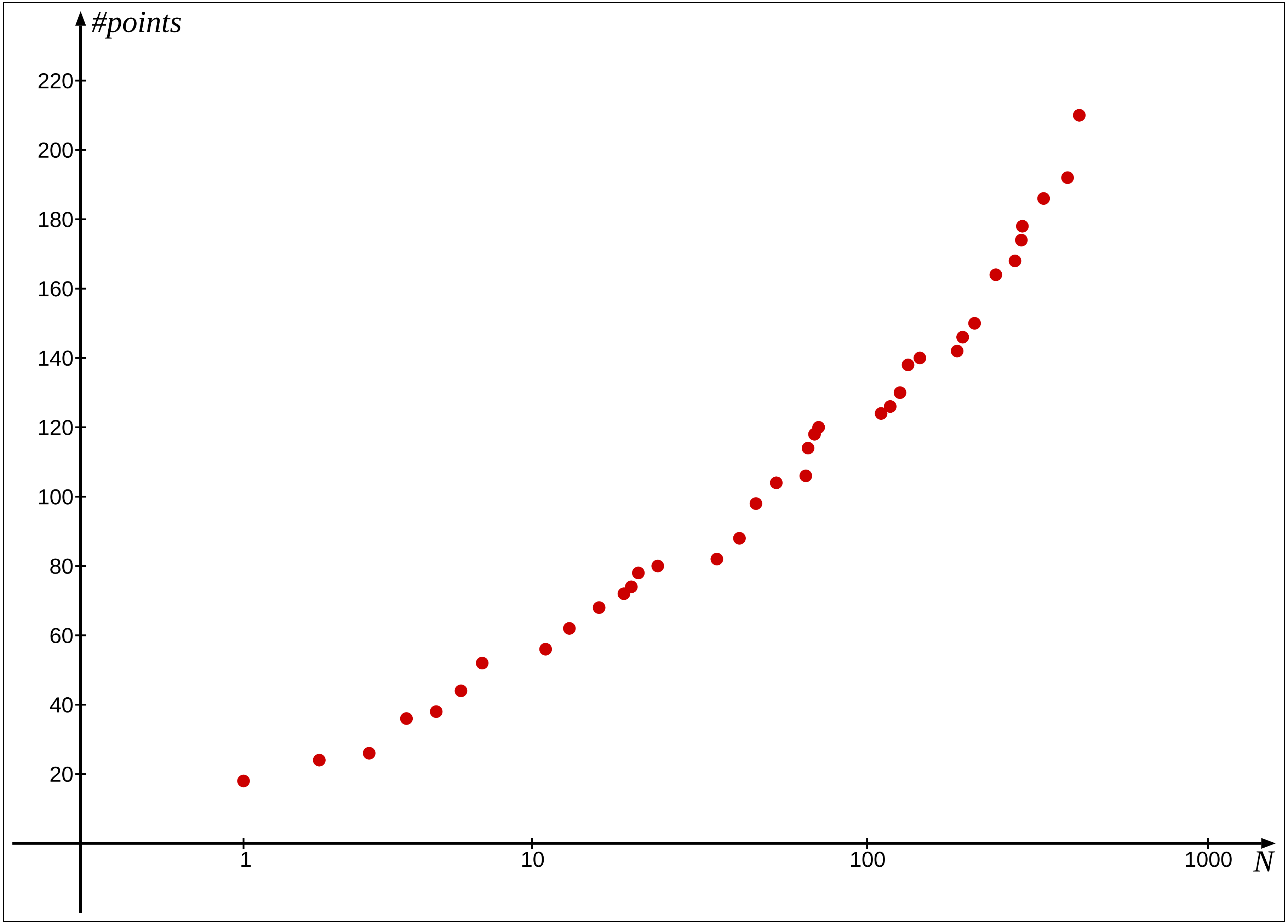
This leads to the following.

**Conjecture 4.**
There is some $B > 0$ such that

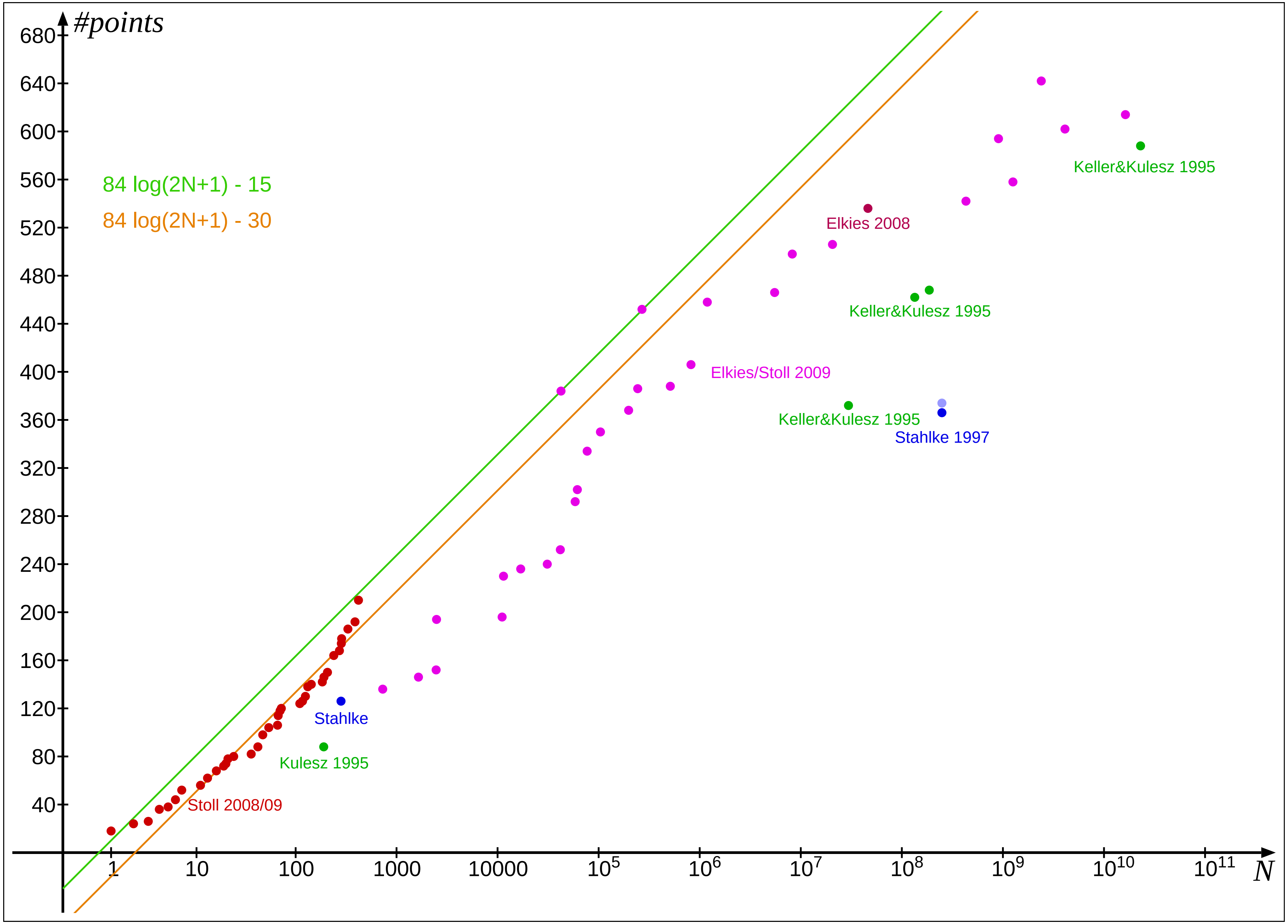
$$\#C(\mathbb{Q}) \leq B \log(2N + 1)$$

for curves of size $\leq N$.

The next two slides show the best curves known to me with respect to number of points versus size.



#points
220
200
180
160
140
120
100
80
60
40
20
1
10
100
1000
N


#points
680
640
600
560
520
480
440
400
360
320
280
240
200
160
120
80
40
84 log(2N+1) - 15
84 log(2N+1) - 30
Keller&Kulesz 1995
Elkies 2008
Keller&Kulesz 1995
Elkies/Stoll 2009
Keller&Kulesz 1995
Stahlke 1997
Stahlke
Kulesz 1995
Stoll 2008/09
1
10
100
1000
10000
$10^5$
$10^6$
$10^7$
$10^8$
$10^9$
$10^{10}$
$10^{11}$
N

# A New Record

The points marked Elkies/Stoll 2009 come from several families constructed by Noam Elkies using K3 surfaces.

One of them represents the current record for the number of rational points on a genus 2 curve over $\mathbb{Q}$:

The curve

$$y^2 = 82342800\,x^6 - 470135160\,x^5 + 52485681\,x^4 + 2396040466\,x^3 + 567207969\,x^2 - 985905640\,x + 247747600$$

has at least **642** rational points!

The previous record was 588 points, due to Keller and Kulesz.

# Some Remarks

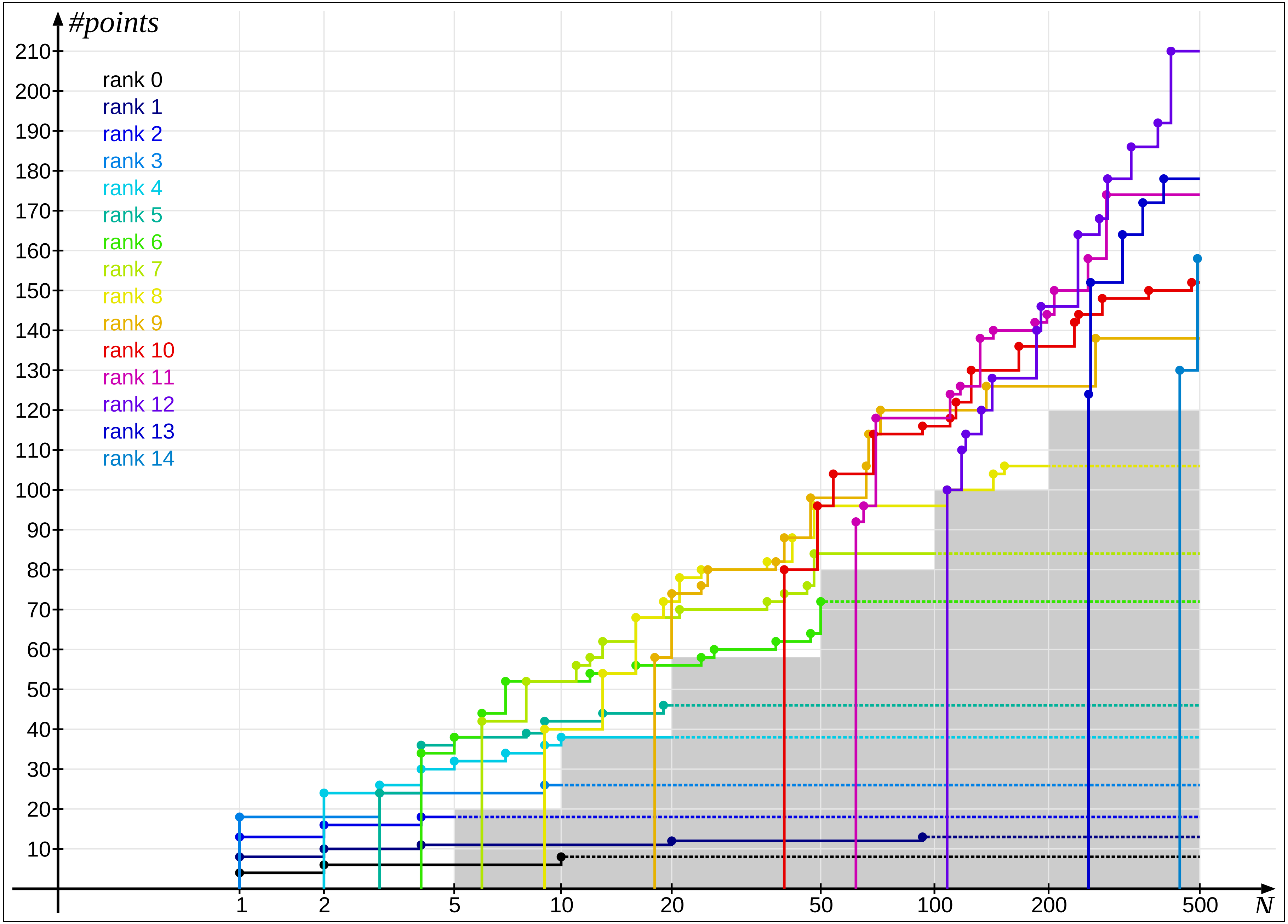
From the data, it looks like we might have

$$\max\{\#C(\mathbb{Q}) : C \text{ of size } \leq N\} \gg \log(2N + 1) \,.$$

However, Caporaso, Harris and Mazur show that
the Weak Lang Conjecture implies that $\#C(\mathbb{Q})$ is bounded.

We observe that the rank of the group of rational points
on the Jacobian variety of the curve also grows
with the number of rational points on the curve.
(Compare next slide.)

So if we assume that the rank is bounded,
our data and the CHM result can be reconciled.



#points
210
200
190
180
170
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
rank 0
rank 1
rank 2
rank 3
rank 4
rank 5
rank 6
rank 7
rank 8
rank 9
rank 10
rank 11
rank 12
rank 13
rank 14
1
2
5
10
20
50
100
200
500
N