Finite Coverings and Rational Points

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1. Introduction

The purpose of this talk is to put forward a conjecture. The background is given by the following

Basic Question.

Given a (smooth projective) curve C over a number field k, can we determine explicitly the set $C(k)$ of rational points?

One possible approach to this is to consider an *unramified covering* $D \xrightarrow{\pi} C$ that is *geometrically Galois*. By standard theory, there are only finitely many *twists* $D_j \xrightarrow{\pi_j} C$ of this covering (up to isomorphism over k) such that D_j has points everywhere locally, and

$$C(k) = \coprod_j \pi_j(D_j(k)) .$$

Moreover, the set of these twists is computable (at least in principle).

In particular, if it turns out that there are *no* such twists, then this proves that $C(k)$ is empty (like in the example above). More generally, in this way, we obtain restrictions on the possible location of rational points inside the adelic points of C.

2. The Conjecture

Let me now state a conjecture that essentially says that this approach provides all the information that it possibly can.

Let us define a *residue class* on C to be a subset X of the adelic points $C(\mathbb{A}_k) = \prod_v C(k_v)$ of the form

$$X = \prod_{v \in S} X_v \times \prod_{v \notin S} C(k_v)$$

with a finite set S of places, where X_v is an open and closed subset of $C(k_v)$.

There will be two versions of the conjecture, a weaker and a stronger one.

Main Conjecture (weak version).

If $X \subset C(\mathbb{A}_k)$ is a residue class such that $X \cap C(k) = \emptyset$, then there exists an unramified covering $D \xrightarrow{\pi} C$ such that for all twists $D_j \xrightarrow{\pi_j} C$, we have $\pi_j(D_j(\mathbb{A}_k)) \cap X = \emptyset$.

In other words, we can actually *prove* that $X \cap C(k) = \emptyset$ using some unramified covering.

Main Conjecture (strong version).

Same as before, but we require the unramified covering $D \to C$ to be abelian.

Here are some consequences.

- The weak version implies that we can decide if $C(k) = \emptyset$: we search for a point by day and run through the coverings by night (they can be enumerated), until one of the two attacks is successful.
- When $C(k)$ is empty, the strong version is equivalent to saying that the Brauer-Manin obstruction is the only obstruction against rational points on C.

3. Evidence

Now I want to give some evidence for these conjectures.
First a few general facts.

- The strong conjecture is true for curves of genus zero. (Use Hasse Principle and weak approximation.)
- Let C be a curve of genus 1, with Jacobian E. If C represents an element of $Ш(k, E)$ that is not divisible, then the strong conjecture is true for C. It is true for E if and only if the divisible subgroup of $Ш(k, E)$ is trivial.
- Similarly, if C is of genus ≥ 2 and Pic^1_C is a non-divisible element in $Ш(k, J)$ (where J is the Jacobian of C), then the strong conjecture holds for C.
- If $C \to A$ is a nonconstant morphism into an abelian variety A such that $A(k)$ is finite and $Ш(k, A)_{\mathrm{div}} = 0$, then the strong conjecture is true for C. (Stoll, partial results by Colliot-Thélène and Siksek in the context of the Brauer-Manin obstruction)
- Bjorn Poonen has heuristic arguments supporting an even stronger version of the conjecture in case $C(k)$ is empty.

From this and by other means, we get a number of concrete examples.

- The strong conjecture is true for all modular curves $X_0(N)$, $X_1(N)$ and $X(N)$ over $\mathbb{Q}$. (Use Mazur and W. Stein's tables)
- Computations have shown the strong conjecture to hold for all but 1488 genus 2 curves of the form $y^2 = f(x)$, where f has integral coefficients of absolute value at most 3, such that the curve does not have a rational point (here $k = \mathbb{Q}$). Under the assumption that $Ш(k, J)_{\mathrm{div}} = 0$ for the Jacobian J of such a curve, the strong conjecture holds for 1383 out of these 1488 curves. Assuming in addition the Birch and Swinnerton-Dyer conjecture (plus standard conjectures on L-series), the strong conjecture holds for 42 of the remaining 105 curves. We hope to be able to deal with the other 63 curves in due course. (Bruin, Stoll)
- Successful Chabauty computations verify the strong conjecture for residue classes defined in terms of just one place v.

There are also some relative statements that allow us to conclude that some version of the conjecture holds for one curve, if we know it for one or more other curves.

- If either version of the conjecture holds for C/K, where K/k is a finite extension, and $C(K)$ is finite, then it holds for C/k. (Stoll)

- If $C(k)$ is finite and $D \to C$ is a nonconstant morphism, and either version of the conjecture holds for C, then it also holds for D. (Stoll, partial result by Colliot-Thélène in the context of the Brauer-Manin obstruction)
- If $D \to C$ is an unramified covering, $C(k)$ is finite, and the weak version of the conjecture holds for all twists D_j, then it also holds for C. (Stoll)

This allows us to show that one of the two versions holds for a given curve in many cases.

We can also use these results to prove a statement of a somewhat different flavor.

- If the weak conjecture holds for $y^2 = x^6 + 1$ over all number fields k, then it also holds for all hyperelliptic curves of genus ≥ 2 (and many more, perhaps all curves with $g \geq 2$) over any number field. (Use Bogomolov-Tschinkel)

4. More Conjectures

Let me state two more rather plausible conjectures.

"Strong Chabauty" Conjecture.

Assume that $C \to A$ is a nonconstant morphism into an abelian variety such that the image of C is not contained in a proper abelian subvariety. Also assume that $\operatorname{rank} A(k) \leq \dim A - 2$. *Then there is a set of places v of k of density 1 and a zero-dimensional subscheme $Z \subset C$ such that $C(k_v)$ intersects the topological closure of $J(k)$ in $J(k_v)$ only in points from Z.*

The motivation for this conjecture comes from the fact that in this situation, the system of equations for the intersection is overdetermined. Hence you do not expect solutions unless there is a good reason for them.

- If C satisfies assumptions and conclusion of the above conjecture, and $Ш(k, A)_{\text{div}} = 0$, then the strong version of the main conjecture is true for C. (Stoll)

"Eventually Small Rank" Conjecture.

Let C be a curve of genus ≥ 2. Then there is some $n \geq 1$ such that for all twists D_j of the multiplication-by-n covering of C with $D_j(\mathbb{A}_k) \neq \emptyset$, the Jacobian of D_j has a factor A such that $\operatorname{rank} A(k) \leq \dim A - 2$.

Since the genus of the D_j grows rapidly with n, this essentially says that one does not expect Mordell-Weil ranks to be large compared to the dimension.

- Assume
  1. $Ш(k, A)_{\text{div}} = 0$ for all abelian varieties,
  2. the "Strong Chabauty" conjecture,
  3. the "Eventually Small Rank" conjecture.

  Then the weak version of the main conjecture holds for all curves over k, and $C(k)$ can be determined.

References

[1] M. Stoll, *Finite descent and rational points on curves*, Preprint (2005).