

THE VALUATION OF THE DISCRIMINANT OF A HYPERSURFACE

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ABSTRACT. Let R be a discrete valuation ring, with valuation $v: R \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and residue field k . Let H be a hypersurface $\text{Proj}(R[x_0, \dots, x_n]/\langle f \rangle)$. Let H_k be the special fiber, and let $(H_k)_{\text{sing}}$ be its singular subscheme. Let $\Delta(f)$ be the discriminant of f . We use Zariski's main theorem and degeneration arguments to prove that $v(\Delta(f)) = 1$ if and only if H is regular and $(H_k)_{\text{sing}}$ consists of a nondegenerate double point over k . We also give lower bounds on $v(\Delta(f))$ when H_k has multiple singularities or a positive-dimensional singularity.

1. INTRODUCTION

Throughout this paper, R denotes a discrete valuation ring, with valuation $v: R \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$, maximal ideal $\mathfrak{m} = (\pi)$, and residue field k (except in a few places where k denotes an arbitrary field).

Let $E \subset \mathbb{P}_R^2$ be defined by a Weierstrass equation, with generic fiber an elliptic curve. If the discriminant of the equation has valuation 1, then E is regular and the singular locus of its special fiber consists of a node; this follows from Tate's algorithm [Tat75], for example; see also [Sil94, Lemma IV.9.5(a)]. Our first theorem (Theorem 1.1) generalizes this to hypersurfaces of arbitrary degree and dimension (terminology will be explained later):

Theorem 1.1. *Let $f \in R[x_0, \dots, x_n]$ be a homogeneous polynomial. Let $\Delta(f)$ be its discriminant. Let $H = \text{Proj}(R[x_0, \dots, x_n]/\langle f \rangle)$. Then the following are equivalent:*

- (i) $v(\Delta(f)) = 1$;
- (ii) H is regular, and $(H_k)_{\text{sing}}$ consists of a single nondegenerate double point in $H(k)$.

For hypersurfaces with more than one singularity, we have the following:

Theorem 1.2. *Let f and H be as in Theorem 1.1.*

- (a) *If $(H_k)_{\text{sing}}$ consists of r closed points, $v(\Delta(f)) \geq r$ (Theorem 7.1).*
- (b) *We have $v(\Delta(f)) \geq \dim(H_k)_{\text{sing}} + 1$ (Theorem 11.5(a)).*
- (c) *If $\dim(H_k)_{\text{sing}} \geq 1$, then $v(\Delta(f)) \geq \lfloor (\deg f - 1)/2 \rfloor$ (Theorem 11.5(b)).*

To prove (c), we show that H_k is a limit of hypersurfaces whose singular subscheme is finite but with many points, and we combine this and an argument using restriction of scalars in the equal characteristic case and the Greenberg functor in the mixed characteristic case.

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The paper is organized as follows. Section 2 defines the discriminant Δ of a projective hypersurface $f = 0$ and proves some basic properties of it. Section 3 describes quadratic forms over a discrete valuation ring, and computes their discriminants. Section 4 defines nondegenerate and ordinary double points. Section 5 adapts the proof of the Bertini smoothness theorem to prove that the singular locus of the general singular hypersurface over a field consists of a single ordinary double point. Our proofs require some ingredients from commutative algebra, provided in Section 6. Section 7 proves Theorem 1.2(a) = Theorem 7.1. Section 8 analyzes the minimum valuation of values of a multivariable polynomial on a residue disk, and Section 9 applies this analysis to Δ , viewed as a polynomial in the coefficients of f . Finally, Section 10 proves Theorem 1.1, and Section 11 proves the rest of Theorem 1.2.

2. THE DISCRIMINANT

Fix $n \geq 1$ and $d \geq 2$. Let A be a ring. Let $A[x_0, \dots, x_n]_d$ be the set of homogeneous polynomials of degree d . Let $f \in A[x_0, \dots, x_n]_d$. Let $H = H_f = \text{Proj}(k[x_0, \dots, x_n]/\langle f \rangle) \subset \mathbb{P}_A^n$. Define the **relative singular subscheme** H_{sing} as the closed subscheme of $\mathbb{P}_A^n$ defined by $f = \partial f / \partial x_0 = \dots = \partial f / \partial x_n = 0$. Its complement $H_{\text{smooth}} := H - H_{\text{sing}}$ is the locus of points at which $H \rightarrow \text{Spec } A$ is smooth of relative dimension $n - 1$.

Let $x^{\mathbf{i}}$ range over the $N := \binom{n+d}{n}$ monomials in $\mathbb{Z}[x_0, \dots, x_n]_d$, and let $a_{\mathbf{i}}$ be independent indeterminates in $\mathbb{Z}[\{a_{\mathbf{i}}\}]$, so $F := \sum_{\mathbf{i}} a_{\mathbf{i}} x^{\mathbf{i}}$ is the generic degree d homogeneous polynomial in $x_0, \dots, x_n$. Then the affine space $\mathbb{A}^N := \text{Spec } \mathbb{Z}[\{a_{\mathbf{i}}\}]$ may be viewed as a moduli space for hypersurfaces (one could also remove the origin, or projectivize as in [Sai12, §2.4]). Specializing the previous paragraph to $f = F$ and $A = \mathbb{Z}[\{a_{\mathbf{i}}\}]$ gives the **universal hypersurface** $\mathcal{H} \subset \mathbb{P}^n \times \mathbb{A}^N$ and its relative singular subscheme $\mathcal{H}_{\text{sing}}$, relative to the second projection $\phi: \mathcal{H} \rightarrow \mathbb{A}^N$.

The first projection makes $\mathcal{H}_{\text{sing}} \rightarrow \mathbb{P}^n$ a rank $N - n - 1$ vector bundle since the equations $F = \partial F / \partial x_0 = \dots = \partial F / \partial x_n = 0$ are linear in the $a_{\mathbf{i}}$ and independent above each point of $\mathbb{P}^n$ except for the Euler relation $d \cdot F = \sum x_i (\partial F / \partial x_i)$. Thus $\mathcal{H}_{\text{sing}}$ is integral and smooth of relative dimension $N - 1$ over $\mathbb{Z}$. Since ϕ is proper, the image $D := \phi(\mathcal{H}_{\text{sing}}) \subset \mathbb{A}^N$ is a closed integral subscheme; D is the locus parametrizing *singular* hypersurfaces. In fact, $D \subset \mathbb{A}^N$ is a divisor and the restriction $\mathcal{H}_{\text{sing}} \rightarrow D$ of ϕ is birational (see Proposition 5.2 below); this is a Bertini-type statement saying essentially that among hypersurfaces singular at a point, most have singular subscheme consisting of just that point. Thus $D \subset \mathbb{A}^N$ is the zero locus of some polynomial $\Delta \in \mathbb{Z}[\{a_{\mathbf{i}}\}]$ determined up to a unit, i.e., up to sign; Δ is called the **discriminant**. (See [GKZ08, Dem12, Sai12] for other descriptions of Δ .) By definition, if the $a_{\mathbf{i}}$ are specialized to elements of a field k , the resulting hypersurface in $\mathbb{P}_k^n$ is singular (not smooth of dimension $n - 1$) if and only if Δ specializes to 0 in k . It is a classical fact that the polynomial Δ is homogeneous of degree $(n + 1)(d - 1)^n$ in the N variables [EH16, Proposition 7.4].

3. QUADRATIC FORMS

Proposition 3.1. *Suppose that $d = 2$. Let $\text{Det} = \det(\partial^2 F / \partial x_i \partial x_j) \in \mathbb{Z}[\{a_{\mathbf{i}}\}]$. If n is odd, then $\Delta = \pm \text{Det}$. If n is even, then $\Delta = \pm \text{Det} / 2$.*

Proof. This is well known, except perhaps the power of 2, which can be determined by evaluating Det for a quadratic form defining a smooth quadric over $\mathbb{Z}$, since $\Delta = \pm 1$ for such a form. Use $x_0 x_1 + \dots + x_{n-1} x_n$ if n is odd, and $x_0 x_1 + \dots + x_{n-2} x_{n-1} + x_n^2$ if n is even. $\square$

Let R be a discrete valuation ring or field. A **quadratic space** over R is a pair (M, q) where M is a finite-rank free R -module (since R is a PID, we do not to say the word *projective*), and $q: M \rightarrow R$ is such that if $e_1, \dots, e_n$ is a basis of M , then $q(x_1e_1 + \dots + x_ne_n)$ is given by a polynomial in $R[x_1, \dots, x_n]_2$. A **symmetric bilinear space** over R is a pair (M, β) where M is as before and $\beta: M \times M \rightarrow R$ is a symmetric R -bilinear pairing. Given q , define $\beta(x, y) := q(x + y) - q(x) - q(y)$; in this way, every quadratic space has an associated symmetric bilinear space.

Proposition 3.2. *Let k be an algebraically closed field. A general quadratic form $q \in k[x_1, \dots, x_n]_2$ is equivalent (via a linear change of variables) to*

$$\begin{cases} x_1^2 + \dots + x_n^2, & \text{if } \text{char } k \neq 2; \\ x_1x_2 + x_3x_4 + \dots + x_{n-1}x_n, & \text{if } \text{char } k = 2 \text{ and } n \text{ is even;} \\ x_1x_2 + x_3x_4 + \dots + x_{n-2}x_{n-1} + x_n^2, & \text{if } \text{char } k = 2 \text{ and } n \text{ is odd.} \end{cases}$$

(General means that there is a dense open subset U of the coefficient space such that the statement holds for q corresponding to a point of U .)

Proof. The associated symmetric bilinear space may be identified with the matrix $M := (\partial^2 q / \partial x_i \partial x_j)$.

First suppose that $\text{char } k \neq 2$. For the general q , the symmetric matrix M has rank n (since $\det M \neq 0$ defines a nonempty open subset), and after a change of variables M is diagonal and q is $x_1^2 + \dots + x_n^2$.

Next suppose that $\text{char } k = 2$. Then M is symplectic, so rank M is even. If n is even, then for the general q , the matrix M is of rank n , and after a change of variable to put M in standard form, q is $x_1x_2 + \dots + x_{n-1}x_n$. Now suppose that n is odd. For the general q , the matrix M is of rank $n - 1$. After a change of variables, q is $x_1x_2 + \dots + x_{n-2}x_{n-1} + \ell^2$, for some linear form ℓ . By adding a multiple of x_1 to x_2 , we may assume that x_1 does not appear in ℓ . Similarly, we can eliminate $x_2, \dots, x_{n-1}$ from ℓ , so ℓ is a multiple of x_n . For the general q , we may assume that ℓ is a *nonzero* multiple of x_n . By scaling, we may assume that $\ell = x_n$. Now $q = x_1x_2 + x_3x_4 + \dots + x_{n-2}x_{n-1} + x_n^2$. $\square$

Proposition 3.3. *Let R be a discrete valuation ring.*

- (a) *Each symmetric bilinear space over R is an orthogonal direct sum of spaces of rank 1 or 2.*
- (b) *Every quadratic form $f(x_0, \dots, x_n)$ over R is equivalent to one of the form*

$$\sum_{i=1}^I (a_i x_i^2 + b_i x_i y_i + c_i y_i^2) + \sum_{j=1}^J d_j z_j^2$$

with $2I + J = n + 1$ and $a_i, b_i, c_i, d_j \in R$.

- (c) *Let f be as in (b). Let $H = \text{Proj}(R[x_0, \dots, x_n]/\langle f \rangle)$. If H_k is smooth, then $v(\Delta(f)) = 0$. Otherwise, $v(\Delta(f)) \geq \dim(H_k)_{\text{sing}} + 1$.*

Proof.

- (a) (We paraphrase an argument of Jean-Pierre Tignol adapted from the proof of [Ver19, Proposition 4.10].) Let (M, β) be a nonzero symmetric bilinear space. We may assume that $\beta \neq 0$. By dividing β by a nonzero element of R , we may assume that $\beta(M, M) \not\subset \mathfrak{m}$. We claim that there exists a free R -module N of rank 1 or 2 with a homomorphism $N \rightarrow M$

such that β induces a **regular** pairing on N (i.e., the composition $N \rightarrow M \xrightarrow{\beta} M^\vee \rightarrow N^\vee$ is an isomorphism); then $N \rightarrow M$ is injective, and M is the orthogonal direct sum of N and $N^\perp := \ker(M \rightarrow N^\vee)$, so we are done by induction on $\text{rank}(M)$.

If there exists $e \in M$ with $\beta(e, e) \in R^\times$ a unit, then let $N = Re$. Otherwise, choose $c, d \in M$ with $\beta(c, d) \in R^\times$ and let $N = Rc \oplus Rd$; the induced pairing is regular since its matrix is invertible, being congruent mod π to $\begin{pmatrix} 0 & \beta(c, d) \\ \beta(c, d) & 0 \end{pmatrix}$.

- (b) Decomposing a quadratic space is equivalent to decomposing the associated symmetric bilinear space, even if $\text{char } k = 2$.
- (c) The case where H_k is smooth is true by the definition of discriminant, so suppose that $(H_k)_{\text{sing}} \neq \emptyset$.

First suppose $\text{char } k \neq 2$. Then f is equivalent to $\sum a_i x_i^2$ for some $a_i \in R$, and

$$\dim (H_k)_{\text{sing}} = \#\{i : v(a_i) \geq 1\} - 1 \leq v(\text{Det}(f)) - 1 = v(\Delta(f)) - 1,$$

by Proposition 3.1.

Now suppose $\text{char } k = 2$. Let $I_0 = \#\{i : v(b_i) = 0\}$ and $I_1 = \#\{i : v(b_i) \geq 1\}$. Let $J_0 = \#\{j : v(d_j) = 0\}$ and $J_1 = \#\{j : v(d_j) \geq 1\}$. If n is odd, let $J' := J$. If n is even, let $J' := J - 1$. In both cases $J' \geq 0$ (if n is even, then J is odd). The common zero locus in $\mathbb{P}_k^n$ of the polynomials $\partial f / \partial x_i$ and $\partial f / \partial y_i$ for $i \in I_0$ is of dimension $n - 2I_0$, and including the condition $f = 0$ drops the dimension by 1 more if $J_0 \geq 1$. Thus $\dim (H_k)_{\text{sing}} \leq n - 2I_0$, with strict inequality if $J_0 \geq 1$. On the other hand, $v(4a_i c_i - b_i^2) \geq 2$ whenever $v(b_i) \geq 1$, and $v(2d_j) \geq v(2) + v(d_j)$ for all j , so Proposition 3.1 implies

$$\begin{aligned} v(\Delta(f)) &\geq 2I_1 + J'v(2) + J_1 \\ &= (n - 2I_0) + J'v(2) - J_0 + 1 \\ &\geq \dim (H_k)_{\text{sing}} + J'v(2) - J_0 + 1. \end{aligned}$$

If $J_0 \geq 1$, then the inequality above is strict and $J'v(2) \geq (J_0 - 1)v(2) \geq J_0 - 1$, so $v(\Delta(f)) \geq \dim (H_k)_{\text{sing}} + 1$. If $J_0 = 0$, then instead use $J'v(2) \geq 0$ to again get $v(\Delta(f)) \geq \dim (H_k)_{\text{sing}} + 1$. $\square$

4. NONDEGENERATE DOUBLE POINTS AND ORDINARY DOUBLE POINTS

Definition 4.1 ([SGA 7_I, VI.6]). Let k be a field. Let X be a finite-type k -scheme. A k -point $Q \in X$ is called a **nondegenerate double point** (or **nondegenerate quadratic point**) if there exist $n \geq 1$ and $f \in k[[x_1, \dots, x_n]]$ such that there is an isomorphism of complete k -algebras $\widehat{\mathcal{O}}_{X,Q} \simeq k[[x_1, \dots, x_n]] / \langle f \rangle$ and an equality of ideals $\langle \partial f / \partial x_1, \dots, \partial f / \partial x_n \rangle = \langle x_1, \dots, x_n \rangle$.

Remark 4.2. The ideal equality is equivalent to saying that Q is an isolated reduced point of the singular subscheme X_{sing} .

Remark 4.3. If X is an affine hypersurface in $\mathbb{A}_k^n$ given by the equation $h(x_1, \dots, x_n) = 0$, then a singular point Q on X is a nondegenerate double point if and only if $\det \left(\frac{\partial^2 h}{\partial x_i \partial x_j} \right)$ does not vanish at Q .

Remark 4.4. Suppose that n and f as in Definition 4.1 exist. Then f can be taken to be a quadratic form [SGA 7_I, VI.6.1]. If, moreover, k is algebraically closed, then

- if $\text{char } k \neq 2$, then one can take $f := x_1^2 + \dots + x_n^2$;

- if $\text{char } k = 2$, then n must be even and one can take $f := x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n$.

Definition 4.5 ([SGA 7_I, Definition VI.6.6]). There is also the notion of **ordinary double point**, which is the same except when $\text{char } k = 2$ and the local dimension n of X at Q is odd. In that case, nondegeneracy is impossible so one calls a singularity an ordinary double point if and only if it is analytically equivalent over an algebraic closure to that defined by $x_1x_2 + \cdots + x_{n-2}x_{n-1} + x_n^2$.

5. THE GENERAL SINGULAR LOCUS

Proposition 5.1. *Fix $n \geq 1$ and $d \geq 2$ and an algebraically closed field k . For general $f \in k[x_0, \dots, x_n]_d$ with H_f singular (that is, f corresponding to a general point of $D(k)$), the hypersurface H_f has a unique singularity and it is an ordinary double point.*

Proof. *Case $d = 2$.* Let $M = (\partial^2 f / \partial x_i \partial x_j) \in M_{n+1}(k)$.

First suppose that $\text{char } k \neq 2$. For the general f , the symmetric matrix M has rank n (rank $\geq n$ is an open condition, and rank $n + 1$ would mean that H_f is smooth), and after a change of variable it is diagonal and f is $x_1^2 + \cdots + x_n^2$, and $(H_f)_{\text{sing}}$ is the single reduced point $(1 : 0 : \cdots : 0)$.

Next suppose that $\text{char } k = 2$. Then M is symplectic, so rank M is even. If n is even, then for the general f , the matrix M is of rank n , and after a change of variable to put M in standard form, f is $x_1x_2 + \cdots + x_{n-1}x_n$, and $(H_f)_{\text{sing}}$ is the single reduced point $(1 : 0 : \cdots : 0)$. If n is odd, then for the general f , the matrix M is of rank $n - 1$, and after a change of variable, f is $x_1x_2 + \cdots + x_{n-2}x_{n-1} + x_n^2$, and $(H_f)_{\text{sing}}$ is a nonreduced degree 2 scheme supported at $(1 : 0 : \cdots : 0)$.

In all these cases, the unique point of $(H_f)_{\text{sing}}$ is an ordinary double point of H_f (and it is even nondegenerate, except when $\text{char } k = 2$ and n is odd).

Case $d \geq 3$. Let $(\mathbb{P}^n \times \mathbb{P}^n)'$ be the locus of pairs of points $(P, Q) \in \mathbb{P}^n \times \mathbb{P}^n$ with $P \neq Q$. Let I be the locus of $(f, P, Q) \in \mathbb{A}^N \times (\mathbb{P}^n \times \mathbb{P}^n)'$ such that H_f is singular at both P and Q . The fibers of $I \rightarrow (\mathbb{P}^n \times \mathbb{P}^n)'$ are linear subspaces of codimension $2n + 2$ in $\mathbb{A}^N$ since we may assume $P = (1 : 0 : \cdots : 0)$ and $Q = (0 : \cdots : 0 : 1)$, in which case H_f is singular at P and Q if and only if the coefficients of $x_0^{d-1}x_i$ and $x_n^{d-1}x_i$ for $i = 0, \dots, n$ all vanish. Thus $\dim I = (N - (2n + 2)) + \dim(\mathbb{P}^n \times \mathbb{P}^n)' = N - 2$, so I does not dominate the $(N - 1)$ -dimensional locus $D \subset \mathbb{A}^N$ corresponding to f with H_f singular.

Thus for general f with H_f singular, H_f has only one singularity, which we may assume is $P := (1 : 0 : \cdots : 0)$. Proposition 3.2 applied to the degree 2 Taylor polynomial at P of a dehomogenization of f shows that for general f , the singularity is an ordinary double point. $\square$

We use subscripts to denote base change: e.g., $D_A := D \times_{\text{Spec } \mathbb{Z}} \text{Spec } A$ and $\mathcal{H}_{\text{sing}, A} := \mathcal{H}_{\text{sing}} \times_{\text{Spec } \mathbb{Z}} \text{Spec } A \simeq (\mathcal{H}_A)_{\text{sing}}$ for any ring A . For an irreducible scheme X , let $\kappa(X)$ be the function field of the integral scheme X_{red} . Recall that $\phi: \mathcal{H} \hookrightarrow \mathbb{P}^n \times \mathbb{A}^N \rightarrow \mathbb{A}^N$ was the second projection. Restricting ϕ yields a proper surjective morphism $\varphi: \mathcal{H}_{\text{sing}} \rightarrow D$.

Proposition 5.2. *The morphism $\varphi: \mathcal{H}_{\text{sing}} \rightarrow D$ is birational. The same holds after base change to any integral domain A , except when $\text{char } A = 2$ and n is odd, in which case $\kappa(\mathcal{H}_{\text{sing}, A})$ is purely inseparable of degree 2 over $\kappa(D_A)$.*

Proof. We may assume that A is a field k , and that k is algebraically closed. The result follows from [Sai12, Proposition 2.12], except when $\text{char } k = 2$ and n is odd. We will reprove those cases and prove the missing cases.

For a general $f \in D(k)$, Proposition 5.1 implies that H_f has an ordinary double point, so $(H_f)_{\text{sing}}$ is a finite connected scheme of degree 1 or 2, the latter occurring exactly when $\text{char } k = 2$ and n is odd. Since $(H_f)_{\text{sing}}$ is the fiber of φ above $f \in D(k)$, the general fiber of φ is described as in the previous sentence. The scheme $\mathcal{H}_{\text{sing},k}$ is smooth over k , hence irreducible, and its image under φ is topologically D_k , so D_k is irreducible too. The result follows from the previous two sentences. $\square$

Let $D_{\text{finite}} := \{d \in D : \dim \varphi^{-1}(d) = 0\}$; this is the subset of points such that the corresponding hypersurface has finitely many singular points. Let $D_1 := \{d \in D : \varphi^{-1}(d) \rightarrow \{d\} \text{ is an isomorphism}\}$; this is the subset of points such that the singular locus of the corresponding hypersurface is a single reduced point.

Lemma 5.3.

- (a) *The subsets $D_1 \subset D_{\text{finite}} \subset D$ are open in D . Identify them with open subschemes of D .*
- (b) *$\varphi^{-1}(D_{\text{finite}}) \rightarrow D_{\text{finite}}$ is the normalization of D_{finite} . The same holds after base change to any normal noetherian domain A , except when $\text{char } A = 2$ and n is odd. In the exceptional case, $\varphi^{-1}(D_{\text{finite},A})$ is the normalization of $(D_{\text{finite},A})_{\text{red}}$ in the purely inseparable extension $\kappa(\mathcal{H}_{\text{sing},A})$ of its function field.*
- (c) *$\varphi^{-1}(D_1) \rightarrow D_1$ is an isomorphism of schemes over $\mathbb{Z}$.*

Proof.

- (a) By [EGA IV₃, Corollaire 13.1.5], D_{finite} is open in D . Openness of D_1 will follow from the proof of (c).
- (b) By Proposition 5.2, $\varphi^{-1}(D_{\text{finite}}) \rightarrow D_{\text{finite}}$ is birational. It is also quasi-finite and proper, hence finite by Zariski's main theorem [EGA III₁, Corollaire 4.4.11]. Moreover, $\mathcal{H}_{\text{sing}}$ is smooth over $\mathbb{Z}$, hence normal. The previous three sentences imply that $\varphi^{-1}(D_{\text{finite}}) \rightarrow D_{\text{finite}}$ is the normalization of D_{finite} . The same argument works after base change to any normal noetherian domain, except that in the exceptional case, the function field extension in Proposition 5.2 is purely inseparable of degree 2 instead of 1.
- (c) Apply the following to $\varphi^{-1}(D_{\text{finite}}) \rightarrow D_{\text{finite}}$: If $\psi: X \rightarrow Y$ is a scheme-theoretically-surjective finite morphism of noetherian schemes and $y \in Y$ is such that $\psi^{-1}(y) \simeq \{y\}$, then there exists an open neighborhood $U \subset Y$ of y such that $\psi^{-1}(U) \rightarrow U$ is an isomorphism. To prove this statement, we may assume that $Y = \text{Spec } A$ and $X = \text{Spec } B$, where $A \rightarrow B$ is injective; then U may be taken as the complement of the support of the A -module B/A . $\square$

Remark 5.4. In Corollary 10.2, we will identify D_1 with the smooth locus of D .

6. COMMUTATIVE ALGEBRA

A ring extension $R' \supset R$ is called a **weakly unramified extension** if R' too is a discrete valuation ring and π is also a uniformizer of R' .

Lemma 6.1. *Let R be a discrete valuation ring, with residue field k . For any field extension $k' \supset k$, there exists a weakly unramified extension $R' \supset R$ with residue field k' (i.e., isomorphic to k' as k -algebra).*

Proof. If k'/k is generated by one algebraic element, say a zero of a monic irreducible polynomial $\bar{f} \in k[x]$, then we may take $R' := R[x]/\langle f \rangle$ for any monic $f \in R[x]$ reducing to $\bar{f}$ [Ser79, I.6, Proposition 15]. If k'/k is generated by one transcendental element t , then we may take the localization $R' := R[t]_{(\pi)}$ of the (regular) polynomial ring $R[t]$ at the codimension 1 prime (π) ; the residue field of R' is $\text{Frac}(R[t]/\langle \pi \rangle) = k(t)$. The general case follows from Zorn's lemma, using direct limits. $\square$

Lemma 6.2. *Let A be a noetherian local ring. Let $\hat{A}$ be its completion. Let B be the integral closure of A_{red} (in its fraction field). Then*

$$\#\{\text{minimal primes of } \hat{A}\} \geq \#\{\text{maximal ideals of } B\}.$$

Proof. Combine [SP, Tag 0C24] and [SP, Tag 0C28(1)]. $\square$

The following is well known; see [SP, Tag 0BRA] for a generalization.

Lemma 6.3. *Let B a normal domain. Let $L = \text{Frac } B$. Let L'/L be a purely inseparable extension. Let B' be the integral closure of B in L' . Then $\text{Spec } B' \rightarrow \text{Spec } B$ is a homeomorphism. In particular, B and B' have the same number of maximal ideals.*

Proof. We may assume that $p := \text{char } L > 0$. The map $\text{Spec } B \rightarrow \text{Spec } B'$ sending $\mathfrak{p}$ to $\{x \in L' : x^{p^e} \in \mathfrak{p} \text{ for some } e \geq 0\}$ is an inverse to $\text{Spec } B' \rightarrow \text{Spec } B$. Thus $\text{Spec } B' \rightarrow \text{Spec } B$ is a continuous bijection between quasi-compact spaces, so it is a homeomorphism. The final sentence follows since maximal ideals correspond to closed points. $\square$

Lemma 6.4. *Let $m \geq 1$. Suppose that $\text{char}(k) = \text{char}(R)$. Then $R/\mathfrak{m}^m \simeq k[t]/\langle t^m \rangle$. If R is complete, then $R \simeq k[[t]]$.*

Proof. The ring $R/\mathfrak{m}^m$ is (trivially) a complete local ring as defined in [Coh46]. We have $\text{char}(k) = \text{char}(R/\mathfrak{m}^m)$, so by [Coh46, Thm. 9], k embeds into $R/\mathfrak{m}^m$. Then the surjective homomorphism $k[t] \rightarrow R/\mathfrak{m}^m$ mapping t to π has kernel $\langle t^m \rangle$. Taking inverse limits gives $R \simeq k[[t]]$ if R is complete. $\square$

7. HYPERSURFACES WITH SEVERAL SINGULARITIES

Let $0 \neq f \in R[x_0, \dots, x_n]_d$ and set $H = \text{Proj}(R[x_0, \dots, x_n]/\langle f \rangle)$.

Theorem 7.1. *If the space $(H_k)_{\text{sing}}$ consists of r closed points, then $v(\Delta(f)) \geq r$.*

Proof. The inequality is trivial if $r = 0$, so assume $r > 0$.

Let $P \in D_R(k)$ correspond to H_k , so $\varphi^{-1}(P) = (H_k)_{\text{sing}}$. Since R is regular, the local rings $\mathcal{O}_{\mathbb{A}_R^N, P}$ and $\hat{\mathcal{O}}_{\mathbb{A}_R^N, P}$ are regular too, and hence factorial [AB59, Theorem 5]. Since $\dim(H_k)_{\text{sing}} = 0$, we have $P \in D_{\text{finite}, R}(k)$ (notation as in Section 5). Let $L = \kappa(D_R)$ and $L' = \kappa(\mathcal{H}_{\text{sing}, R})$. By Lemma 5.3, $D_{\text{finite}, R}$ is open in D_R , and $\varphi^{-1}(D_{\text{finite}, R}) \rightarrow (D_{\text{finite}, R})_{\text{red}}$ is the normalization of $(D_{\text{finite}, R})_{\text{red}}$ in the purely inseparable extension L'/L .

Localizing at P on the target, we obtain a morphism $\text{Spec } B' \rightarrow \text{Spec } A_{\text{red}}$, where $A := \mathcal{O}_{D_{\text{finite}, R}, P} = \mathcal{O}_{D_R, P} = \mathcal{O}_{\mathbb{A}_R^N, P}/\langle \Delta \rangle$, and B' is the integral closure of A_{red} in L' . Define $\hat{A}$ and B as in Lemma 6.2, so $\hat{A} \simeq \hat{\mathcal{O}}_{\mathbb{A}_R^N, P}/\langle \Delta \rangle$, and B is the integral closure of A_{red} in L . The maximal ideals of B' correspond to the points of $\varphi^{-1}(D_{\text{finite}, R})$ above P , which are the r points of $(H_k)_{\text{sing}}$. By Lemma 6.3, B too has r maximal ideals. By Lemma 6.2, $\hat{A}$ has at least

r minimal primes. Their inverse images in $\widehat{\mathcal{O}}_{\mathbb{A}_R^N, P}$ correspond to prime factors of Δ in this factorial ring, so $\Delta = p_1 \cdots p_r q$, for some $p_1, \dots, p_r, q \in \widehat{\mathcal{O}}_{\mathbb{A}_R^N, P}$ with each p_i vanishing at P . Evaluation at the coefficient tuple of f defines a ring homomorphism $\widehat{\mathcal{O}}_{\mathbb{A}_R^N, P} \rightarrow \widehat{R}$ sending Δ to $\Delta(f)$ and sending each p_i into the maximal ideal of $\widehat{R}$, so $v(\Delta(f)) \geq 1 + \cdots + 1 + 0 = r$. $\square$

8. VALUATIONS OF POLYNOMIAL VALUES

Lemma 8.1. *Let $\rho: \mathbb{A}_k^\ell \rightarrow \mathbb{A}_k^n$ be a projection for some $\ell \geq n$. Let $V \subset \mathbb{A}_k^\ell$ be a closed subscheme. Then $\{a \in \mathbb{A}_k^n : \rho^{-1}(a) \subseteq V\}$ is closed in $\mathbb{A}_k^n$.*

Proof. Since ρ is flat, ρ is open, so $\rho(\mathbb{A}_k^n - V)$ is open; its complement is closed. $\square$

Definition 8.2. Let $H = \text{Spec}(k[x_1, \dots, x_n]/\langle f \rangle) \subset \mathbb{A}_k^n$ be a hypersurface and let $a \in k^n$. Let $\mathfrak{m}_a$ be the maximal ideal of $k[x_1, \dots, x_n]$ corresponding to a . Then $\text{mult}_H(a)$ denotes the multiplicity of a as a point on H , i.e.,

$$\text{mult}_H(a) = \max\{m \in \mathbb{Z}_{\geq 0} : f \in \mathfrak{m}_a^m\}.$$

For $b \in R$, let $\bar{b}$ be its image in k . Likewise, given $b \in R^n$, define $\bar{b} \in k^n$. Fix a nonzero polynomial $\delta \in R[x_0, \dots, x_n]$. (Eventually δ will be Δ .) From now on, we assume that k is infinite.

Definition 8.3. Define $\text{vmin}_\delta: k^n \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by

$$\text{vmin}_\delta(a) = \min\{v(\delta(b)) : b \in R^n \text{ with } \bar{b} = a\}.$$

Lemma 8.4. *The integer $\min\{v(\delta(b)) : b \in R^n\}$ equals the minimum of the valuations of the coefficients of δ .*

Proof. By dividing by a power of π , we may assume that some coefficient is a unit. The reduction $\bar{\delta} \in k[x_1, \dots, x_n]$ is not the zero polynomial, and k is infinite, so $\bar{\delta}$ is nonvanishing at some point in k^n . Lift the point to $b \in R^n$. Then $v(\delta(b)) = 0$. Thus both minima equal 0. $\square$

Corollary 8.5. *For $b \in R^n$, the integer $\text{vmin}_\delta(\bar{b})$ equals the minimum of the valuations of the coefficients of $\delta(b + \pi x)$.*

Proof. Apply Lemma 8.4 to $\delta(b + \pi x)$. $\square$

Proposition 8.6. *The function vmin_δ on $\mathbb{A}^n(k)$ is upper-semicontinuous with respect to the Zariski topology.*

Proof. We need to show that for $m \in \mathbb{Z}_{\geq 0}$, the set $\{a \in k^n : \text{vmin}_\delta(a) \geq m\}$ is $W(k)$ for some closed subscheme $W \subset \mathbb{A}_k^n$. Let $R_m = R/\mathfrak{m}^m$.

Case 1: R is of equal characteristic. By Lemma 6.4, R_m is a k -algebra of vector space dimension m . Applying restriction of scalars $\text{Res}_{R_m/k}$ to $\delta: \mathbb{A}_{R_m}^n \rightarrow \mathbb{A}_{R_m}^1$ produces a morphism $\mathbb{A}_k^{mn} \rightarrow \mathbb{A}_k^m$; let V be the fiber above 0. The reduction map $R_m^n \rightarrow k^n$ arises from a morphism $\rho: \mathbb{A}_k^{mn} \rightarrow \mathbb{A}_k^n$ that is a projection as in Lemma 8.1. Let W be a closed subscheme whose underlying space is the closed subset of Lemma 8.1. Then for $a \in k^n$, the following are equivalent (note that $\rho^{-1}(a)$ is an affine space):

$$\text{vmin}_\delta(a) \geq m, \quad \rho^{-1}(a)(k) \subset V(k), \quad \rho^{-1}(a) \subset V, \quad a \in W(k).$$

Case 2: R is of mixed characteristic with perfect residue field k . In the previous argument, replace $\text{Res}_{R_m/k}$ with the Greenberg functor Gr^m from R_m -schemes to k -schemes; see [Gre61; Gre63; NS08, §2.2; BGA18].

Case 3: R is of mixed characteristic with imperfect residue field k . Let k' be the perfect closure of k . Use Lemma 6.1 to find a weakly unramified extension $R' \supset R$ with residue field k' . Let $W' = \text{Spec}(k'[x_1, \dots, x_n]/\langle f_1, \dots, f_r \rangle)$ be the closed subscheme for R' in Case 2. By replacing each f_i by $f_i^{p^n}$ for some n , we may assume that $f_i \in k[x_1, \dots, x_n]$, without changing $W'(k')$. Let $W = \text{Spec}(k[x_1, \dots, x_n]/\langle f_1, \dots, f_r \rangle)$. By Corollary 8.5, $\text{vmin}_\delta(a)$ is the same whether we work with R or R' , so $\{a \in k^n : \text{vmin}_\delta(a) \geq m\} = k^n \cap W'(k') = W(k)$. $\square$

Let $V = \text{Spec}(R[x_1, \dots, x_n]/\langle \delta \rangle)$. From now on, assume that some coefficient of δ is a unit, so that V_k is a hypersurface in $\mathbb{A}_k^n$.

Lemma 8.7. *Let $a \in k^n$. Then $\text{vmin}_\delta(a) \leq \text{mult}_{V_k}(a)$.*

Proof. Without loss of generality, $a = 0$. Let $m = \text{mult}_{V_k}(0)$. Some degree m monomial in $\delta(x)$ has a unit coefficient, so some degree m monomial in $\delta(\pi x)$ has valuation m . On the other hand, $\text{vmin}_\delta(0)$ is the minimum of the valuations of $\delta(\pi x)$, so it is at most m . $\square$

Proposition 8.8. *Let $a \in k^n$. Then $\text{vmin}_\delta(a) \geq 2$ if and only if $a \in (V_k)_{\text{sing}}$ and a is in the image of the reduction map $V(R/\mathfrak{m}^2) \rightarrow V(k)$.*

Proof. By shifting, we may assume $a = 0$. Write $\delta(x) = r + \sum_{i=1}^n s_i x_i + \dots$. The following are equivalent:

- $\text{vmin}_\delta(0) \geq 2$;
- the minimum of the valuations of the coefficients of $\delta(\pi x)$ is at least 2 (see Corollary 8.5);
- $v(r) \geq 2$ and $v(s_i) \geq 1$ for all i .

The last conditions imply that $0 \in V(R/\mathfrak{m}^2)$ and $0 \in (V_k)_{\text{sing}}$. Conversely, if $0 \in (V_k)_{\text{sing}}$, then $v(r) \geq 1$ and $v(s_i) \geq 1$ for all i , and if moreover 0 is the image of some $b_2 \in V(R/\mathfrak{m}^2)$, then we may lift b_2 to $b \in (\pi R)^n$ with $v(\delta(b)) \geq 2$, which is equivalent to $v(r) \geq 2$ since $v(s_i) \geq 1$ for all i . $\square$

9. MINIMAL VALUATIONS OF THE DISCRIMINANT

In this section, we assume that k is algebraically closed. Recall the definitions of Δ and D from Section 2. We apply the results of the previous section with $\delta := \Delta$. For $a \in k^N$, let $f_a \in k[x_0, \dots, x_n]_d$ be the polynomial with coefficients given by a , and let $H_a = H_{f_a} \subset \mathbb{P}_k^n$.

By Theorem 7.1, if $(H_a)_{\text{sing}}$ consists of r isolated points, then $\text{vmin}_\Delta(a) \geq r$.

Lemma 9.1. *Fix $b \in R^N$.*

- (a) *Let $V \subset \mathbb{A}_k^N$ be a variety such that $\bar{b} \in V(k)$. If $\{a \in V(k) : \text{vmin}_\Delta(a) \geq m\}$ is Zariski dense in V , then $v(\Delta(b)) \geq m$.*
- (b) *If there exists $a \in k^N$ such that $(H_a)_{\text{sing}}$ is finite and contains r distinct points $P_1, \dots, P_r$ that are also singularities of $H_{\bar{b}}$, then $v(\Delta(b)) \geq r$.*

Proof.

- (a) By Proposition 8.6, $\{a \in V(k) : \text{vmin}_\Delta(a) \geq m\} = V(k) \ni \bar{b}$, so $v(\Delta(b)) \geq m$.

(b) If $\bar{b} = a$, then $v(\Delta(b)) \geq \text{vmin}_\Delta(a) \geq r$ by Theorem 7.1. If $\bar{b} \neq a$, let $V \subset \mathbb{A}_k^N$ be the line joining $\bar{b}$ and a . Since the condition that a given point $P \in \mathbb{P}^n$ is a singular point of a hypersurface H is linear in the coefficients of the polynomial defining H , all points $c \in V(k)$ will have the property that $\{P_1, \dots, P_r\} \subset (H_c)_{\text{sing}}$.

By Lemma 5.3(a), “ $\dim(H_c)_{\text{sing}} \geq 1$ ” is a closed condition, so for all but finitely many $c \in V(k)$, we have $\dim(H_c)_{\text{sing}} = 0$, so $(H_c)_{\text{sing}}$ is a finite set containing $P_1, \dots, P_r$. Theorem 7.1 implies that $\text{vmin}_\Delta(c) \geq r$ for all these points. The claim now follows from part (a). $\square$

Lemma 9.2. *The reduction map $D(R) \rightarrow D(k)$ is surjective.*

Proof. Let $a \in D(k)$; then H_a has a singular point $P \in \mathbb{P}^n(k)$. We may assume that $P = (1 : 0 : \dots : 0)$. The condition that H_a is singular at P is given by the vanishing of certain coordinates. Lift $a \in k^N$ to some $b \in R^N$ so that these coordinates remain zero. $\square$

Corollary 9.3. *Let $a \in k^N$. Then $\text{vmin}_\Delta(a) \geq 2$ if and only if $a \in (D_k)_{\text{sing}}$.*

Proof. By Lemma 9.2, every $a \in D(k)$ is in the image of $D(R/\mathfrak{m}^2)$. Apply Proposition 8.8 to $\delta = \Delta$. $\square$

We now prove a variant of Theorem 7.1, in which the r singularities need not be isolated, but they must be linearly independent.

Lemma 9.4. *Let $P_1, \dots, P_r \in \mathbb{P}^n(k)$ be points that span a $\mathbb{P}^{r-1}$ and let $a_0 \in k^N$ be such that H_{a_0} is singular at $P_1, \dots, P_r$. Then $\text{vmin}_\Delta(a_0) \geq r$ and $\text{mult}_{D_k}(a_0) \geq r$.*

Proof. We can assume the P_j to be coordinate points. Since Δ vanishes on $a \in k^N$ when H_a is singular in P_j , each term in Δ must be divisible by one of the coordinates that describe the vanishing of a and its first partial derivatives at P_j . When the degree d is at least 3, then these sets of coordinates are disjoint in pairs, and so

$$\Delta \in \langle a(P_1), \nabla a(P_1) \rangle \cdots \langle a(P_r), \nabla a(P_r) \rangle.$$

This implies that $\text{mult}_{D_k}(a_0) \geq r$ and also that $\text{vmin}_\Delta(a_0) \geq r$.

The result is still valid when $d = 2$. In this case, the associated reduced subscheme of $(H_{a_0})_{\text{sing}}$ is a linear space, so the $\mathbb{P}^{r-1}$ spanned by $P_1, \dots, P_r$ is contained in $(H_{a_0})_{\text{sing}}$. Then by Proposition 3.3(c), $\text{vmin}_\Delta(a_0) \geq r$. By Lemma 8.7, $\text{mult}_{D_k}(a_0) \geq \text{vmin}_\Delta(a_0) \geq r$. $\square$

Lemma 9.4 generalizes Proposition 3.3(c) to forms of arbitrary degree.

For a subset $X \subset \mathbb{P}^n(k)$, let $\text{Span } X$ be the smallest linear subspace of $\mathbb{P}^n$ containing X . If $X = \emptyset$, use the conventions $\text{Span } X \simeq \mathbb{P}^{-1} = \emptyset$ and $\dim \text{Span } X = -1$.

Corollary 9.5. *Let $b \in R^N$. Then $v(\Delta(b)) \geq \dim \text{Span}((H_{\bar{b}})_{\text{sing}}(k)) + 1$.*

Proof. Let $r = \dim \text{Span}((H_{\bar{b}})_{\text{sing}}(k)) + 1$. Choose $P_1, \dots, P_r \in (H_{\bar{b}})_{\text{sing}}(k)$ that span a $\mathbb{P}^{r-1}$. Now apply Lemma 9.4. $\square$

We will obtain better lower bounds in Section 11.

Corollary 9.6. *Let $b \in R^N$ such that $v(\Delta(b)) = 1$. Then $(H_{\bar{b}})_{\text{sing}}$ consists of a single point.*

Proof. Since $\Delta(\bar{b}) = \overline{\Delta(b)} = 0$, $H_{\bar{b}}$ has at least one singularity. If $(H_{\bar{b}})_{\text{sing}}$ contained at least two points, then $v(\Delta(b)) \geq \text{vmin}_\Delta(\bar{b}) \geq 2$ by Lemma 9.4, contradicting the assumption. $\square$

See Corollary 10.1 below for a more precise statement.

10. WHEN THE DISCRIMINANT HAS VALUATION 1

We now characterize when $v(\Delta(f)) = 1$. Recall the statement from the introduction:

Theorem 1.1. *Let $f \in R[x_0, \dots, x_n]$ be a homogeneous polynomial. Let $\Delta(f)$ be its discriminant. Let $H = \text{Proj}(R[x_0, \dots, x_n]/\langle f \rangle)$. Then the following are equivalent:*

- (i) $v(\Delta(f)) = 1$;
- (ii) H is regular, and $(H_k)_{\text{sing}}$ consists of a single nondegenerate double point in $H(k)$.

Proof. *Case 1: char $k = 2$ and n is odd.* By [Sai12, Theorem 4.2], if the sign of Δ is chosen appropriately, then $\Delta = A^2 + 4B$ for some polynomials A, B , so $v(\Delta(f)) \neq 1$. On the other hand, by Remark 4.4, H_k cannot have a nondegenerate double point. Thus (i) and (ii) both fail.

Case 2: char $k \neq 2$ or n is even. The surjection $R[\{a_i\}] \twoheadrightarrow R$ sending the a_i to the corresponding coefficients α_i of f defines an R -morphism $\iota: \text{Spec } R \rightarrow \mathbb{A}_R^N$. Then $H \rightarrow \text{Spec } R$ is the pullback by ι of $\mathcal{H}_R \rightarrow \mathbb{A}_R^N$. Let $P = \iota(\text{Spec } k) \in \mathbb{A}^N(k)$.

(i) $\Rightarrow$ (ii): Suppose that $v(\Delta(f)) = 1$. By Corollary 9.6, $(H_k)_{\text{sing}}$ consists of a single point. The surjection $R[\{a_i\}] \twoheadrightarrow R$ maps Δ to $\Delta(f)$, so the $a_i - \alpha_i$ and Δ are local parameters for $\mathbb{A}_R^N$ at P . Thus $D_R = \text{Spec}(R[\{a_i\}]/\langle \Delta \rangle)$ is regular at P , so D_R is normal at P . Then Lemma 5.3(b) implies that the fiber $(H_k)_{\text{sing}} = \varphi^{-1}(P)$ consists of a single reduced k -point Q . By Remark 4.2, Q is a nondegenerate double point of H_k .

Choose an $\mathbb{A}_R^n \subset \mathbb{P}_R^n$ containing Q ; let f_0 be the corresponding dehomogenization of f . The point $(H_k)_{\text{sing}}$ is cut out in $\mathbb{A}_R^n$ by f_0 and its partial derivatives; these $n+1$ functions are therefore local parameters for $\mathbb{P}_R^n$ at Q , so the local ring $\mathcal{O}_{H,Q} = \mathcal{O}_{\mathbb{P}_R^n,Q}/\langle f_0 \rangle$ is regular too. On the other hand, $H - \{Q\}$ is smooth over $\text{Spec } R$. Thus H is regular everywhere.

(ii) $\Rightarrow$ (i): Now suppose that H is regular and $(H_k)_{\text{sing}}$ consists of a nondegenerate double point $Q \in H(k)$. Let f_0 be as above, so f_0 and its partial derivatives lie in the maximal ideal $\mathfrak{m}_{\mathbb{P}_R^n,Q} \subset \mathcal{O}_{\mathbb{P}_R^n,Q}$. Since Q is a nondegenerate double point, the partial derivatives form a basis for $\mathfrak{m}_{\mathbb{P}_k^n,Q}/\mathfrak{m}_{\mathbb{P}_k^n,Q}^2$, so they are independent in $\mathfrak{m}_{\mathbb{P}_R^n,Q}/\mathfrak{m}_{\mathbb{P}_R^n,Q}^2$. On the other hand, the image of f_0 in $\mathfrak{m}_{\mathbb{P}_R^n,Q}/\mathfrak{m}_{\mathbb{P}_R^n,Q}^2$ is nonzero (since $\mathcal{O}_{H,Q} = \mathcal{O}_{\mathbb{P}_R^n,Q}/\langle f_0 \rangle$ is regular) and in fact *independent* of the partial derivatives (since it maps to 0 in $\mathfrak{m}_{\mathbb{P}_k^n,Q}/\mathfrak{m}_{\mathbb{P}_k^n,Q}^2$). Thus f_0 and its partial derivatives form a basis of $\mathfrak{m}_{\mathbb{P}_R^n,Q}/\mathfrak{m}_{\mathbb{P}_R^n,Q}^2$, so by Nakayama's lemma, they generate $\mathfrak{m}_{\mathbb{P}_R^n,Q}$, so $H_{\text{sing}} \simeq \text{Spec } k$.

Pulling back $(\mathcal{H}_R)_{\text{sing}} \rightarrow D_R \hookrightarrow \mathbb{A}_R^N$ by ι gives $H_{\text{sing}} \rightarrow \text{Spec}(R/\langle \Delta(f) \rangle) \rightarrow \text{Spec } R$. Since $(H_k)_{\text{sing}}$ is a single reduced k -point, $P \in D_1(k)$. By Lemma 5.3(a), $D_{1,R}$ is open in D_R , so $\text{Spec}(R/\langle \Delta(f) \rangle)$ is contained in $D_{1,R}$. By Lemma 5.3(c), $(\mathcal{H}_R)_{\text{sing}} \rightarrow D_R$ is an isomorphism above $D_{1,R}$, so $H_{\text{sing}} \simeq \text{Spec}(R/\langle \Delta(f) \rangle)$. By the previous paragraph, $H_{\text{sing}} \simeq \text{Spec } k$, so $v(\Delta(f)) = 1$. $\square$

Corollary 10.1. *Assume that k is algebraically closed. For $a \in k^N$ (and for every choice of R with residue field k), the following statements are equivalent.*

- (a) $\text{vmin}_\Delta(a) = 1$.
- (b) a is a smooth point on D_k .
- (c) $(H_a)_{\text{sing}}$ consists of a single nondegenerate double point.
- (d) $a \in D_1(k)$.

Proof.

(a) $\Leftrightarrow$ (b): The following are equivalent: $\text{vmin}_\Delta(a) > 0$; $\Delta(a) = 0$ in k ; $a \in D(k)$. By Corollary 9.3, $\text{vmin}_\Delta(a) \geq 2$ if and only if $a \in D_{\text{sing}}(k)$.

(a) $\Rightarrow$ (c): Use Theorem 1.1.

(c) $\Rightarrow$ (a): Let $a \in k^N$ be such that $(H_a)_{\text{sing}}$ consists of a single nondegenerate double point Q . Lift a to $b \in R^N$. Then H_b is regular at every point of its special fiber except possibly Q . By adding a multiple of π to b if necessary, we may assume that H_b is regular also at Q . The regular locus of H_b is open and contains the special fiber, so H_b is regular. Theorem 1.1 applied to H_b implies that $v(\Delta(b)) = 1$. On the other hand, if b' is any lift of a , then $v(\Delta(b')) \geq 1$ since H_a is singular. Thus $\text{vmin}_\Delta(a) = 1$.

(c) $\Leftrightarrow$ (d): Use Remark 4.2 and the definition of D_1 . $\square$

Corollary 10.2. *The subscheme D_1 is the smooth locus of $D \rightarrow \text{Spec } \mathbb{Z}$.*

Proof. This can be checked on geometric points, and every field k is the residue field of some discrete valuation ring R . Apply Corollary 10.1(b) $\Leftrightarrow$ (d). $\square$

11. HYPERSURFACES WITH A POSITIVE-DIMENSIONAL SINGULARITY

In Lemma 11.1, Corollary 11.3, and Lemma 11.4, we assume that $n \geq 2$, $r \geq 1$, and $P_1, \dots, P_r$ are distinct points in $\mathbb{P}^n(k)$. Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}^n_k}$. For each $P \in \mathbb{P}^n(k)$, let $\mathfrak{m}_P \subset \mathcal{O}$ be the ideal sheaf of P .

Lemma 11.1. *If $d \geq 2r - 1$, then $\mathcal{O}(d) \rightarrow \prod_i (\mathcal{O}/\mathfrak{m}_{P_i}^2)(d)$ induces a surjection on global sections.*

Proof. Surjectivity of a linear map is unchanged by field extension, so we may assume that k is infinite. Then we can choose, for each $1 \leq i \leq r$, a linear form ℓ_i vanishing at P_i but not at P_j for any $j \neq i$. We can also choose a homogeneous polynomial h of degree $d - (2r - 1)$ not vanishing at any P_i . For each s , as g ranges over linear forms, the image of g in $(\mathcal{O}/\mathfrak{m}_{P_s}^2)(1)$ ranges over all its sections, so the images of $gh \prod_{j \neq s} \ell_j^2$ in $\prod_i (\mathcal{O}/\mathfrak{m}_{P_i}^2)(d)$ exhaust the s th factor of $\prod_i (\mathcal{O}/\mathfrak{m}_{P_i}^2)(d)$. $\square$

Remark 11.2. The result of Lemma 11.1 is sharp when the points $P_1, \dots, P_r$ are on a line: in this case, no $d < 2r - 1$ will have the stated property.

Recall the definitions of N and H_f from Section 2. Let $Z \subset \mathbb{A}^N$ be the subvariety whose points correspond to f such that $(H_f)_{\text{sing}}$ contains $P_1, \dots, P_r$.

Corollary 11.3. *If $d \geq 2r - 1$, then Z is an affine space of dimension $N - r(n + 1)$.*

Proof. The set $Z(k)$ is the kernel of the surjection in Lemma 11.1. $\square$

Lemma 11.4. *Assume that k is algebraically closed. If $r \leq (d - 1)/2$, then there exists $f \in k[x_0, \dots, x_n]_d$ such that $(H_f)_{\text{sing}} = \{P_1, \dots, P_r\}$ as a set.*

Proof. Let $P = \{P_1, \dots, P_r\}$. Let

$$I = \{(f, P_{r+1}) \in Z \times (\mathbb{P}^n - P) : P_{r+1} \in (H_f)_{\text{sing}}\}.$$

The fiber of $I \rightarrow \mathbb{P}^n - P$ above P_{r+1} consists of the f for which $(H_f)_{\text{sing}} \supset \{P_1, \dots, P_{r+1}\}$, so its dimension is $N - (r + 1)(n + 1)$ by Corollary 11.3, which also implies $\dim Z = N - r(n + 1)$. Thus $\dim I = \dim(\mathbb{P}^n - P) + N - (r + 1)(n + 1) = \dim Z - 1$. Therefore there exists a point in Z outside the image of I ; this defines f . $\square$

Theorem 11.5. *Let $n, d \geq 2$. Let $f \in R[x_0, \dots, x_n]_d$. Let $H = H_f$. Assume that $\dim(H_k)_{\text{sing}} \geq 1$.*

- (a) $v(\Delta(f)) \geq \dim(H_k)_{\text{sing}} + 1 \geq 2$.
- (b) $v(\Delta(f)) \geq \lfloor (d-1)/2 \rfloor$.
- (c) *If $n = 2$, then $v(\Delta(f)) \geq 2d - 3$ if $d \neq 4$ and $v(\Delta(f)) \geq 4$ if $d = 4$.*
- (d) *If $(H_k)_{\text{sing}}$ contains a line, then $v(\Delta(f)) \geq d - 1$.*

Proof. Using Lemma 6.1, we may reduce to the case in which k is algebraically closed.

- (a) This follows from Corollary 9.5.
- (b) Let $r = \lfloor (d-1)/2 \rfloor$. Choose distinct points $P_1, \dots, P_r \in (H_k)_{\text{sing}}$. By Lemma 11.4, there exists $h \in k[x_0, \dots, x_n]_d$ such that $(H_h)_{\text{sing}} = \{P_1, \dots, P_r\}$ as a set. By Lemma 9.1(b), $v(\Delta(f)) \geq r$ as claimed.
- (c) In the case $n = 2$ of plane curves, $\bar{f} = g^2h$ for some g of some degree m with $1 \leq m \leq d/2$ and h of degree $d - 2m$. In Lemma 9.1(a) we take V to be the variety consisting of all forms factoring as g_1g_2h with $\deg g_1 = \deg g_2 = m$ and $\deg h = d - 2m$; then $\bar{f} \in V(k)$. Let $V' \subset V$ be the dense open subvariety defined by the additional conditions that g_1, g_2, h define smooth curves intersecting transversely. By Bézout's theorem, if $a \in V'(k)$, then $\#(H_a)_{\text{sing}} = m^2 + 2m(d - 2m)$, so $\text{vmin}_\Delta(a) \geq m^2 + 2m(d - 2m)$ by Theorem 7.1. Lemma 9.1(a) then shows that $v(\Delta(f)) \geq m^2 + 2m(d - 2m)$. The bound in the statement is obtained by taking the minimum over m in $[1, d/2]$. For $d \neq 4$, the minimum is obtained for $m = 1$; when $d = 4$, $m = 2$ gives the smaller value.
- (d) Let L be a line contained in $(H_k)_{\text{sing}}$. We will construct an auxiliary polynomial $h \in k[x_0, \dots, x_n]_d$ such that $(H_h)_{\text{sing}}$ is finite and contains $d - 1$ points on L . We may assume that L is $x_2 = x_3 = \dots = x_n = 0$. Choose distinct $c_1, \dots, c_{d-1} \in k$. Let $g \in k[x_3, \dots, x_n]_d$ such that $H_g \subset \mathbb{P}^{n-3}$ is smooth; if $n = 2$, then $g = 0$. Let $h = x_2 \prod_{i=1}^{d-1} (x_1 - c_i x_0) + g(x_3, \dots, x_n)$.
Suppose that $Q \in (H_h)_{\text{sing}}$. At Q , we have $\partial h / \partial x_2 = 0$ so $\prod_{i=1}^{d-1} (x_1 - c_i x_0) = 0$; also $h = 0$, so $g = 0$; also, $\partial g / \partial x_i = 0$ for $i = 3, \dots, n$, but H_g is smooth. Thus $x_3 = \dots = x_n = 0$ at Q , and Q is a singular point on the union of lines $x_2 \prod_{i=1}^{d-1} (x_1 - c_i x_0) = 0$ in $\mathbb{P}^2$, hence $(0 : 0 : 1)$ or $(1 : c_i : 0)$ for some i . Thus $(H_h)_{\text{sing}}$ is finite and contains $d - 1$ points on L . Lemma 9.1(b) gives $v(\Delta(f)) \geq d - 1$. $\square$

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REFERENCES

- [AB59] Maurice Auslander and D. A. Buchsbaum, *Unique factorization in regular local rings*, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 733–734, DOI 10.1073/pnas.45.5.733. MR103906 $\uparrow 7$
- [BGA18] Alessandra Bertapelle and Cristian D. González-Avilés, *The Greenberg functor revisited*, Eur. J. Math. **4** (2018), no. 4, 1340–1389, DOI 10.1007/s40879-017-0210-0. MR3866700 $\uparrow 9$
- [Coh46] I. S. Cohen, *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc. **59** (1946), 54–106, DOI 10.2307/1990313. MR16094 $\uparrow 7$
- [Dem12] Michel Demazure, *Résultant, discriminant*, Enseign. Math. (2) **58** (2012), no. 3-4, 333–373 (French). MR3058604 $\uparrow 2$

- [EGA III₁] A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. **11** (1961). Written in collaboration with J. Dieudonné. MR0217085 (36 #177c) ↑[6](#)
- [EGA IV₃] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966). Written in collaboration with J. Dieudonné. MR0217086 (36 #178) ↑[6](#)
- [EH16] David Eisenbud and Joe Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge University Press, Cambridge, 2016. MR3617981 ↑[2](#)
- [GKZ08] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2008. Reprint of the 1994 edition. MR2394437 (2009a:14065) ↑[2](#)
- [Gre61] Marvin J. Greenberg, *Schemata over local rings*, Ann. of Math. (2) **73** (1961), 624–648, DOI 10.2307/1970321. MR126449 ↑[9](#)
- [Gre63] Marvin J. Greenberg, *Schemata over local rings. II*, Ann. of Math. (2) **78** (1963), 256–266, DOI 10.2307/1970342. MR156855 ↑[9](#)
- [NS08] Johannes Nicaise and Julien Sebag, *Motivic Serre invariants and Weil restriction*, J. Algebra **319** (2008), no. 4, 1585–1610, DOI 10.1016/j.jalgebra.2007.11.006. MR2383059 ↑[9](#)
- [Sai12] Takeshi Saito, *The discriminant and the determinant of a hypersurface of even dimension*, Math. Res. Lett. **19** (2012), no. 4, 855–871, DOI 10.4310/MRL.2012.v19.n4.a10. MR3008420 ↑[2](#), [6](#), [11](#)
- [Ser79] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg. MR554237 (82e:12016) ↑[7](#)
- [SGA 7_I] *Groupes de monodromie en géométrie algébrique. I*, Lecture Notes in Mathematics, vol. 288, Springer-Verlag, Berlin, 1972 (French). Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I); Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim. MR0354656 (50 #7134) ↑[4](#), [5](#)
- [Sil94] Joseph H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 151, Springer-Verlag, New York, 1994. MR96b:11074 ↑[1](#)
- [SP] The Stacks Project authors, *Stacks project*, October 15, 2025. Available at <http://stacks.math.columbia.edu>. ↑[7](#)
- [Tat75] J. Tate, *Algorithm for determining the type of a singular fiber in an elliptic pencil*, Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1975, pp. 33–52. Lecture Notes in Math., Vol. 476. MR0393039 ↑[1](#)
- [Ver19] Joachim Verstraete, *Arason’s filtration of the Witt group of dyadic valued fields*, J. Algebra **519** (2019), 190–227, DOI 10.1016/j.jalgebra.2018.10.026. MR3876186 ↑[3](#)

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