FINITE DESCENT OBSTRUCTIONS AND RATIONAL POINTS ON CURVES

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Abstract. Let k be a number field and X a smooth projective k-variety. In this paper, we discuss the information obtainable from descent via torsors under finite k-group schemes on the location of the k-rational points on X within the adelic points. We relate finite descent obstructions to the Brauer-Manin obstruction; in particular, we prove that on curves, the Brauer set equals the set cut out by finite abelian descent. We show that on many curves of genus ≥ 2, the information coming from finite abelian descent cuts out precisely the rational points. If such a curve fails to have rational points, this is then explained by the Brauer-Manin obstruction. In the final section of this paper, we present some conjectures that are motivated by our results.

Contents

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1. Introduction

In this paper we explore what can be deduced about the set of rational points on a curve (or a more general variety) from a knowledge of its finite étale coverings.

Given a smooth projective variety X over a number field k and a finite étale, geometrically Galois covering $\pi : Y \to X$, standard descent theory tells us that there are only finitely many twists $\pi_j : Y_j \to X$ of π such that Y_j has points everywhere locally, and then $X(k) = \coprod_j \pi_j(Y_j(k))$. Since $X(k)$ embeds into the adelic points $X(\mathbb{A}_k)$, we obtain restrictions on where the rational points on X can be located inside $X(\mathbb{A}_k)$: we must have

$$X(k) \subset \bigcup_j \pi_j(D_j(\mathbb{A}_k)) =: X(\mathbb{A}_k)^{\pi} .$$

Taking the information from all such finite étale coverings together, we arrive at

$$X(\mathbb{A}_k)^{\text{f-cov}} = \bigcap_{\pi} X(\mathbb{A}_k)^{\pi} .$$

Since the information we get cannot tell us more than on which connected component a point lies at the infinite places, we make a slight modification by replacing the v-adic component of $X(\mathbb{A}_k)$ with its set of connected components, for infinite places v. In this way, we obtain $X(\mathbb{A}_k)_\bullet$ and $X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$.

We can be more restrictive in the kind of coverings we allow. We denote the set cut out by restrictions coming from finite abelian coverings only by $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ and the set cut out by solvable coverings by $X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$. Then we have the chain of inclusions

$$X(k) \subset \overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-sol}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet ,$$

where $\overline{X(k)}$ is the topological closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet$, see Section 5 below.

One important result of this paper is that the set cut out by the information coming from finite étale abelian coverings on a curve C coincides with the 'Brauer set', which is defined using the Brauer group of C:

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(\mathbb{A}_k)_\bullet^{\text{Br}}$$

This is shown in Section 7. In this way, it becomes possible to study the Brauer-Manin obstruction on curves via finite étale abelian coverings. For example, we provide an alternative proof of the main result in Scharaschkin's thesis [Sc] characterizing $C(\mathbb{A}_k)_\bullet^{\text{Br}}$ in terms of the topological closure of the Mordell-Weil group in the adelic points of the Jacobian, see Cor. 7.4

Let us call X "good" if it satisfies $\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ and "very good" if it satisfies $\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$.

Then another consequence is that the Brauer-Manin obstruction is the only obstruction against rational points on a curve that is very good. More precisely, the Brauer-Manin obstruction is the only one against a weak form of weak approximation, i.e., weak approximation with information at the infinite primes reduced to connected components.

We prove that an abelian variety A/k is very good if and only if the divisible subgroup of $Ш(k, A)$ is trivial. For example, if $A/\mathbb{Q}$ is a modular abelian variety of analytic rank zero, then $A(k)$ is finite, and A is very good. A principal homogeneous space X for A such that $X(k) = \emptyset$ is very good if and only if it represents a non-divisible element of $Ш(k, A)$. See Cor. 6.2 and the text following it.

Furthermore, if C/k is a curve that has a nonconstant morphism $C \to X$, where X is (very) good and $X(k)$ is finite, then C is (very) good (and $C(k)$ is finite), see Prop. 8.5. This implies that every curve $C/\mathbb{Q}$ whose Jacobian has a nontrivial factor A that is a modular abelian variety of analytic rank zero is very good, see Cor. 8.6. As an application, we prove that all modular curves $X_0(N)$, $X_1(N)$ and $X(N)$ (over $\mathbb{Q}$) are very good, see Cor. 8.8. For curves without rational points, we have the following corollary:

If $C/\mathbb{Q}$ has a non-constant morphism into a modular abelian variety of analytic rank zero, and if $C(\mathbb{Q}) = \emptyset$, then the absence of rational points is explained by the Brauer-Manin obstruction.

This generalizes a result due to Siksek [Si] by removing all assumptions related to the Galois action on the fibers of the morphism over rational points.

In a second, shorter part of the paper, which makes up Section 9, we use our results as a motivation to conjecture that all curves over number fields are very good, and we provide some more evidence for this conjecture.

Since compatible systems of coverings correspond to sections of the canonical homomorphism $\pi_1(X) \to \operatorname{Gal}(\bar{k}/k)$, our work has some bearing on the "Section Conjecture" in anabelian geometry. We show that a curve of genus at least two that has the "section property" is good, see Thm. 9.16. We also show that if the curve is good, then it has the "birational section property", see Thm. 9.18. Since we can prove that certain curves over $\mathbb{Q}$ are good, we obtain a number of examples of curves over $\mathbb{Q}$ that have the birational section property.

The paper is organized as follows. After a preliminary section setting up notation, we prove some results on abelian varieties, which will be needed later on, but are also interesting in themselves, in Section 3. Then, in Section 4, we review torsors and twists and set up some categories of torsors for later use. Section 5 introduces the sets cut out by finite descent information, as sketched above, and Section 6 relates this to rational points. Next we study the relationship between our sets $X(\mathbb{A}_k)_\bullet^{\text{f-cov/f-sol/f-ab}}$ and the Brauer set $X(\mathbb{A}_k)_\bullet^{\text{Br}}$ and its variants. This is done in

Section 7. We then discuss certain inheritance properties of the notion of being "excellent" (which is stronger than "good") in Section 8. This is then the basis for the conjectures formulated in Section 9.

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2. Preliminaries

In all of this paper, k is a number field.

Let X be a smooth projective variety over k. We modify the definition of the set of adelic points of X in the following way.[1]

$$X(\mathbb{A}_k)_\bullet = \prod_{v\nmid\infty} X(k_v) \times \prod_{v\mid\infty} \pi_0(X(k_v)) \,.$$

In other words, the factors at infinite places v are reduced to the set of connected components of $X(k_v)$. We then have a canonical surjection $X(\mathbb{A}_k) \longrightarrow\!\!\!\!\!\rightarrow X(\mathbb{A}_k)_\bullet$. Note that for a zero-dimensional variety (or reduced finite scheme) Z, we have $Z(\mathbb{A}_k) = Z(\mathbb{A}_k)_\bullet$. We will occasionally be a bit sloppy in our notation, pretending that canonical maps like $X(\mathbb{A}_k)_\bullet \to X(\mathbb{A}_K)_\bullet$ (for a finite extension $K \supset k$) or $Y(\mathbb{A}_k)_\bullet \to X(\mathbb{A}_k)_\bullet$ (for a subvariety $Y \subset X$) are inclusions, even though they in general are not at the infinite places. So for example, the intersection $X(K) \cap X(\mathbb{A}_k)_\bullet$ means the intersection of the images of both sets in $X(\mathbb{A}_K)_\bullet$.

If $X = A$ is an abelian variety over k, then

$$\prod_{v\nmid\infty}\{0\} \times \prod_{v\mid\infty} A(k_v)^0 = A(\mathbb{A}_k)_{\mathrm{div}}$$

is exactly the divisible subgroup of $A(\mathbb{A}_k)$. This implies that

$$A(\mathbb{A}_k)_\bullet / nA(\mathbb{A}_k)_\bullet = A(\mathbb{A}_k)/nA(\mathbb{A}_k)$$

and then that

$$A(\mathbb{A}_k)_\bullet = \varprojlim A(\mathbb{A}_k)_\bullet / nA(\mathbb{A}_k)_\bullet = \varprojlim A(\mathbb{A}_k)/nA(\mathbb{A}_k) = \widehat{A(\mathbb{A}_k)}$$

[1] This notation was introduced by Bjorn Poonen.

is (isomorphic to) its own component-wise pro-finite completion and also the component-wise pro-finite completion of the usual group of adelic points.

We will denote by $\widehat{A(k)} = A(k) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ the pro-finite completion $\varprojlim A(k)/nA(k)$ of the Mordell-Weil group $A(k)$. By a result of Serre's [Se1, Thm. 3], the natural map $\widehat{A(k)} \to \widehat{A(\mathbb{A}_k)} = A(\mathbb{A}_k)_\bullet$ is an injection and therefore induces an isomorphism with the topological closure $\overline{A(k)}$ of $A(k)$ in $A(\mathbb{A}_k)_\bullet$. We will re-prove this in Prop. 3.7 below, and even show something stronger than that, see Thm. 3.10. (Our proof is based on a later result of Serre's.) Note that we have an exact sequence

$$0 \longrightarrow A(k)_{\mathrm{tors}} \longrightarrow \widehat{A(k)} \longrightarrow \hat{\mathbb{Z}}^r \longrightarrow 0\,,$$

where r is the Mordell-Weil rank of $A(k)$; in particular,

$$\widehat{A(k)}_{\mathrm{tors}} = A(k)_{\mathrm{tors}}\,.$$

Let $\mathrm{Sel}^{(n)}(k, A)$ denote the n-Selmer group of A over k, as usual sitting in an exact sequence

$$0 \longrightarrow A(k)/nA(k) \longrightarrow \mathrm{Sel}^{(n)}(k, A) \longrightarrow Ш(k, A)[n] \longrightarrow 0\,.$$

If $n \mid N$, we have a canonical map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(k)/NA(k) & \longrightarrow & \mathrm{Sel}^{(N)}(k, A) & \longrightarrow & Ш(k, A)[N] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \cdot N/n} & & \\ 0 & \longrightarrow & A(k)/nA(k) & \longrightarrow & \mathrm{Sel}^{(n)}(k, A) & \longrightarrow & Ш(k, A)[n] & \longrightarrow & 0 \end{array}$$

and we can form the projective limit

$$\widehat{\mathrm{Sel}}(k, A) = \varprojlim \mathrm{Sel}^{(n)}(k, A)\,,$$

which sits again in an exact sequence

$$0 \longrightarrow \widehat{A(k)} \longrightarrow \widehat{\mathrm{Sel}}(k, A) \longrightarrow T\,Ш(k, A) \longrightarrow 0\,,$$

where $TШ(k, A)$ is the Tate module of $Ш(k, A)$ (and exactness on the right follows from the fact that the maps $A(k)/NA(k) \to A(k)/nA(k)$ are surjective). If $Ш(k, A)$ is finite, or more generally, if the divisible subgroup $Ш(k, A)_{\mathrm{div}}$ is trivial, then the Tate module vanishes, and $\widehat{\mathrm{Sel}}(k, A) = \widehat{A(k)}$. Note also that since $T\,Ш(k, A)$ is torsion-free, we have

$$\widehat{\mathrm{Sel}}(k, A)_{\mathrm{tors}} = \widehat{A(k)}_{\mathrm{tors}} = A(k)_{\mathrm{tors}}\,.$$

By definition of the Selmer group, we get maps

$$\mathrm{Sel}^{(n)}(k, A) \longrightarrow A(\mathbb{A}_k)/nA(\mathbb{A}_k) = A(\mathbb{A}_k)_\bullet/nA(\mathbb{A}_k)_\bullet$$

that are compatible with the projective limit, so we obtain a canonical map

$$\widehat{\mathrm{Sel}}(k, A) \longrightarrow A(\mathbb{A}_k)_\bullet$$

through which the map $\widehat{A(k)} \to A(\mathbb{A}_k)_\bullet$ factors. We will denote elements of $\widehat{\mathrm{Sel}}(k, A)$ by $\hat{P}$, $\hat{Q}$ and the like, and we will write P_v, Q_v etc. for their images in $A(k_v)$ or $\pi_0(A(k_v))$, so that the map $\widehat{\mathrm{Sel}}(k, A) \to A(\mathbb{A}_k)_\bullet$ is written $\hat{P} \longmapsto (P_v)_v$. (It will turn out that this map is injective, see Prop. 3.7.)

If X is a k-variety, then we use notation like Pic_X, NS_X, etc., to denote the Picard group, Néron-Severi group, etc., of X over $\bar{k}$, as a k-Galois module.

3. Some results on abelian varieties

In the following, A is an abelian variety over k of dimension g. For $N \geq 1$, we set $k_N = k(A[N])$ for the N-division field, and $k_\infty = \bigcup_N k_N$ for the division field.

The following lemma, based on a result of Serre's on the image of the Galois group in $\mathrm{Aut}(A_{\mathrm{tors}})$, forms the basis for the results of this section.

Lemma 3.1. *There is some $m \geq 1$ such that m kills all the cohomology groups $H^1(k_N/k, A[N])$.*

Proof: By a result of Serre's [Se2, p. 60], the image of G_k in $\mathrm{Aut}(A_{\mathrm{tors}}) = \mathrm{GL}_{2g}(\hat{\mathbb{Z}})$ meets the scalars $\hat{\mathbb{Z}}^\times$ in a subgroup containing $S = (\hat{\mathbb{Z}}^\times)^d$ for some $d \geq 1$. We can ssume that d is even.

Now we note that in

$$H^1(k_N/k, A[N]) \stackrel{\mathrm{inf}}{\hookrightarrow} H^1(k_\infty/k, A[N]) \longrightarrow H^1(k_\infty/k, A_{\mathrm{tors}}),$$

the kernel of the second map is killed by $\#A(k)_{\mathrm{tors}}$. Hence it suffices to show that $H^1(k_\infty/k, A_{\mathrm{tors}})$ is killed by some m.

Let $G = \mathrm{Gal}(k_\infty/k) \subset \mathrm{GL}_{2g}(\hat{\mathbb{Z}})$, then $S \subset G$ is a normal subgroup. We have the inflation-restriction sequence

$$H^1(G/S, A_{\mathrm{tors}}^S) \longrightarrow H^1(G, A_{\mathrm{tors}}) \longrightarrow H^1(S, A_{\mathrm{tors}}).$$

Therefore it suffices to show that there is some integer $D \geq 1$ killing both A_{tors}^S and $H^1(S, A_{\mathrm{tors}}) = H^1((\hat{\mathbb{Z}}^\times)^d, \mathbb{Q}/\mathbb{Z})^{2g}$.

For a prime p, we define

$$\nu_p = \min\{v_p(a^d - 1) : a \in \mathbb{Z}_p^\times\}.$$

It is easy to see that when p is odd, we have $\nu_p = 0$ if $p - 1$ does not divide d, and $\nu_p = 1 + v_p(d)$ otherwise. Also, $\nu_2 = 1$ if d is odd (which we excluded), and

$\nu_2 = 2 + v_2(d)$ otherwise. In particular,

$$D = \prod_p p^{\nu_p}$$

is a well-defined positive integer.

We first show that A^S_{tors} is killed by D. We have

$$A^S_{\text{tors}} = \Bigl(\bigoplus_p (\mathbb{Q}_p/\mathbb{Z}_p)^{(\mathbb{Z}_p^\times)^d}\Bigr)^{2g},$$

and for an individual summand, we see that

$$\begin{aligned}(\mathbb{Q}_p/\mathbb{Z}_p)^{(\mathbb{Z}_p^\times)^d} &= \{x \in \mathbb{Q}_p/\mathbb{Z}_p : (a^d - 1)x = 0 \quad \forall a \in \mathbb{Z}_p^\times\} \\ &= \{x \in \mathbb{Q}_p/\mathbb{Z}_p : p^{\nu_p} x = 0\}\end{aligned}$$

is killed by p^{ν_p}, whence the claim.

Now we have to look at $H^1(S, A_{\text{tors}})$. It suffices to consider $H^1((\hat{\mathbb{Z}}^\times)^d, \mathbb{Q}/\mathbb{Z})$. We start with

$$H^1((\mathbb{Z}_p^\times)^d, \mathbb{Q}_p/\mathbb{Z}_p) = 0\,.$$

Too see this, note that $(\mathbb{Z}_p^\times)^d$ is pro-cyclic (for odd p, $\mathbb{Z}_p^\times$ is already pro-cyclic, for $p = 2$, $\mathbb{Z}_2^\times$ is $\{\pm 1\}$ times a pro-cyclic group, and the first factor goes away under exponentiation by d, since d was assumed to be even); let $\alpha \in (\mathbb{Z}_p^\times)^d$ be a topological generator. By evaluating cocycles at α, we obtain an injection

$$H^1((\mathbb{Z}_p^\times)^d, \mathbb{Q}_p/\mathbb{Z}_p) \hookrightarrow \frac{\mathbb{Q}_p/\mathbb{Z}_p}{(\alpha - 1)(\mathbb{Q}_p/\mathbb{Z}_p)} = \frac{\mathbb{Q}_p/\mathbb{Z}_p}{p^{\nu_p}(\mathbb{Q}_p/\mathbb{Z}_p)} = 0\,.$$

We then can conclude that $H^1((\hat{\mathbb{Z}}^\times)^d, \mathbb{Q}_p/\mathbb{Z}_p)$ is killed by p^{ν_p}. To see this, write

$$(\hat{\mathbb{Z}}^\times)^d = (\mathbb{Z}_p^\times)^d \times T\,,$$

where $T = \prod_{q \neq p} (\mathbb{Z}_q^\times)^d$. Then, by inflation-restriction again, there is an exact sequence

$$0 = H^1((\mathbb{Z}_p^\times)^d, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^1((\hat{\mathbb{Z}}^\times)^d, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^1(T, \mathbb{Q}_p/\mathbb{Z}_p)^{(\mathbb{Z}_p^\times)^d},$$

and we have

$$H^1(T, \mathbb{Q}_p/\mathbb{Z}_p)^{(\mathbb{Z}_p^\times)^d} = \operatorname{Hom}(T, (\mathbb{Q}_p/\mathbb{Z}_p)^{(\mathbb{Z}_p^\times)^d})\,.$$

This group is killed by p^{ν_p}, since $(\mathbb{Q}_p/\mathbb{Z}_p)^{(\mathbb{Z}_p^\times)^d}$ is. It follows that

$$H^1((\hat{\mathbb{Z}}^\times)^d, \mathbb{Q}/\mathbb{Z}) = \bigoplus_p H^1((\hat{\mathbb{Z}}^\times)^d, \mathbb{Q}_p/\mathbb{Z}_p)$$

is killed by $D = \prod_p p^{\nu_p}$.

We therefore find that $H^1(G, A_{\text{tors}})$ is killed by D^2, and that $H^1(k_N/k, A[N])$ is killed by $D^2 \# A(k)_{\text{tors}}$, for all N. □

Remark 3.2. A similar statement is proved for elliptic curves in [Vi, Prop. 7].

Lemma 3.3. *For all positive integers N, the map*

$$Sel^{(N)}(k, A) \longrightarrow \mathrm{Sel}^{(N)}(k_N, A)$$

has kernel killed by m, where m is the number from Lemma 3.1.

Proof: We have the following commutative and exact diagram.

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \ker & \longrightarrow & H^1(k_N/k, A[N]) \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{inf}} \\
0 & \longrightarrow & \mathrm{Sel}^{(N)}(k, A) & \longrightarrow & H^1(k, A[N]) \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\
0 & \longrightarrow & \mathrm{Sel}^{(N)}(k_N, A) & \longrightarrow & H^1(k_N, A[N])
\end{array}$$

So the kernel in question injects into $H^1(k_N/k, A[N])$, and by Lemma 3.1, this group is killed by m. □

Lemma 3.4. *Let $Q \in \mathrm{Sel}^{(N)}(k, A)$, and let n be the order of mQ, where m is the number from Lemma 3.1. Then the density of places v of k such that v splits completely in k_N/k and such that the image of Q in $A(k_v)/NA(k_v)$ is trivial is at most $1/(n[k_N : k])$.*

Proof: By Lemma 3.3, the kernel of $\mathrm{Sel}^{(N)}(k, A) \to \mathrm{Sel}^{(N)}(k_N, A)$ is killed by m. Hence the order of the image of Q in $\mathrm{Sel}^{(N)}(k_N, A)$ is a multiple of n, the order of mQ. Now consider the following diagram for a place v that splits in k_N and a place w of k_N above it.

$$\begin{array}{ccccccc}
\mathrm{Sel}^{(N)}(k, A) & \longrightarrow & \mathrm{Sel}^{(N)}(k_N, A) & \hookrightarrow & H^1(k_N, A[N]) & = & \mathrm{Hom}(G_{k_N}, A[N]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A(k_v)/NA(k_v) & \xrightarrow{\cong} & A(k_{N,w})/NA(k_{N,w}) & \hookrightarrow & H^1(k_{N,w}, A[N]) & = & \mathrm{Hom}(G_{k_{N,w}}, A[N])
\end{array}$$

Let α be the image of Q in $\mathrm{Hom}(G_{k_N}, A[N])$. Then the image of Q is trivial in $A(k_v)/NA(k_v)$ if and only if α restricts to the zero homomorphism on $G_{k_{N,w}}$. This is equivalent to saying that w splits completely in L/k_N, where L is the fixed field of the kernel of α. Since the order of α is a multiple of n, we have $[L : k_N] \geq n$, and the claim now follows from the Chebotarev Density Theorem. □

Recall the definition of $\widehat{\mathrm{Sel}}(k, A)$ and the natural maps

$$A(k) \hookrightarrow \widehat{A(k)} \hookrightarrow \widehat{\mathrm{Sel}}(k, A) \longrightarrow A(\mathbb{A}_k)_\bullet \,,$$

where we denote the rightmost map by

$$\hat{P} \longmapsto (P_v)_v \,.$$

Also recall that $\widehat{\mathrm{Sel}}(k, A)_{\mathrm{tors}} = A(k)_{\mathrm{tors}}$ under the identification given by the inclusions above.

Lemma 3.5. *Let $\hat{Q}_1, \dots, \hat{Q}_s \in \widehat{\mathrm{Sel}}(k, A)$ be elements of infinite order, and let $n \ge 1$. Then there is some N such that the images of $\hat{Q}_1, \dots, \hat{Q}_s$ in $\mathrm{Sel}^{(N)}(k, A)$ all have order at least n.*

Proof: For a fixed $1 \le j \le s$, consider $(n-1)!\hat{Q}_j \neq 0$. There is some N_j such that the image of $(n-1)!\hat{Q}_j$ in $\mathrm{Sel}^{(N_j)}(k, A)$ is non-zero. This implies that the image of $\hat{Q}_j$ has order at least n. Because of the canonical maps $\mathrm{Sel}^{(lN_j)}(k, A) \to \mathrm{Sel}^{(N_j)}(k, A)$, this will also be true for all multiples of N_j. Therefore, any N that is a common multiple of all the N_j will do. □

Proposition 3.6. *Let $Z \subset A$ be a finite subscheme of an abelian variety A over k such that $Z(k) = Z(\bar{k})$. Let $\hat{P} \in \widehat{\mathrm{Sel}}(k, A)$ be such that $P_v \in Z(k_v) = Z(k)$ for a set of places v of k of density 1. Then $\hat{P}$ is in the image of $Z(k)$ in $\widehat{\mathrm{Sel}}(k, A)$.*

Proof: We first show that $\hat{P} \in Z(k) + A(k)_{\mathrm{tors}}$. (Here and in the following, we identify $A(k)$ with its image in $\widehat{\mathrm{Sel}}(k, A)$.) Assume the contrary. Then none of the differences $\hat{P} - Q$ for $Q \in Z(k)$ has finite order. Let $n > \#Z(k)$, then by Lemma 3.5, we can find a number N such that the image of $m(\hat{P} - Q)$ under $\widehat{\mathrm{Sel}}(k, A) \to \mathrm{Sel}^{(N)}(k, A)$ has order at least n, for all $Q \in Z(k)$.

By Lemma 3.3, the density of places of k such that v splits in k_N and at least one of $\hat{P} - Q$ (for $Q \in Z(k)$) maps trivially into $A(k_v)/NA(k_v)$ is at most

$$\frac{\#Z(k)}{n[k_N : k]} < \frac{1}{[k_N : k]} \,.$$

Therefore, there is a set of places v of k of positive density such that v splits completely in k_N/k and such that none of $\hat{P} - Q$ maps trivially into $A(k_v)/NA(k_v)$. This implies $P_v \neq Q$ for all $Q \in Z(k)$, contrary to the assumption on $\hat{P}$ and the fact that $Z(k_v) = Z(k)$.

It therefore follows that $\hat{P} \in Z(k) + A(k)_{\mathrm{tors}} \subset A(k)$. Take a finite place v of k such that $P_v \in Z(k)$ (the set of such places has density 1 by assumption). Then $A(k)$ injects into $A(k_v)$. But the image P_v of $\hat{P}$ under $\widehat{\mathrm{Sel}}(k, A) \to A(k_v)$ is in Z, therefore we must have $\hat{P} \in Z(k)$. □

The following is a simple, but useful consequence.

Proposition 3.7. *If S is a set of places of k of density 1, then*
$$\widehat{\operatorname{Sel}}(k, A) \longrightarrow \prod_{v \in S} A(k_v)/A(k_v)^0$$
is injective. (Note that $A(k_v)^0 = 0$ for v finite.) In particular,
$$\widehat{A(k)} \longrightarrow \prod_{v \in S} A(k_v)/A(k_v)^0$$
is injective, and the canonical map $\widehat{A(k)} \to A(\mathbb{A}_k)_\bullet$ induces an isomorphism between $\widehat{A(k)}$ and $\overline{A(k)}$, the topological closure of $A(k)$ in $A(\mathbb{A}_k)_\bullet$.

This is essentially Serre's result in [Se1, Thm. 3].

Proof: Let $\hat{P}$ be in the kernel. Then we can apply Prop. 3.6 with $Z = \{0\}$, and we find that $\hat{P} = 0$.

In the last statement, it is clear that the image of the map is $\overline{A(k)}$, whence the result. □

From now on, we will identify $\widehat{\operatorname{Sel}}(k, A)$ with its image in $A(\mathbb{A}_k)_\bullet$. We then have a chain of inclusions
$$A(k) \subset \overline{A(k)} \subset \widehat{\operatorname{Sel}}(k, A) \subset A(\mathbb{A}_k)_\bullet\,,$$
and
$$\widehat{\operatorname{Sel}}(k, A)/\overline{A(k)} \cong T\text{Ш}(k, A)$$
vanishes if and only if the divisible subgroup of $\text{Ш}(k, A)$ is trivial.

We can prove a stronger result than the above. For a finite place v of k, we denote by $\mathbb{F}_v$ the residue class field at v. If v is a place of good reduction for A, then it makes sense to speak of $A(\mathbb{F}_v)$, the group of $\mathbb{F}_v$-points of A. There is a canonical map
$$\widehat{\operatorname{Sel}}(k, A) \longrightarrow A(k_v) \longrightarrow A(\mathbb{F}_v)\,.$$

Lemma 3.8. *Let $0 \neq \hat{Q} \in \widehat{\operatorname{Sel}}(k, A)$. Then there is a set of (finite) places v of k (of good reduction for A) of positive density such that the image of $\hat{Q}$ in $A(\mathbb{F}_v)$ is non-trivial.*

Proof: First assume that $\hat{Q} \notin A(k)_{\text{tors}}$. Then $m\hat{Q} \neq 0$, so there is some N such that $m\hat{Q}$ has nontrivial image in $\operatorname{Sel}^{(N)}(k, A)$ (where m is, as usual, the number from Lemma 3.1). By Lemma 3.4, we find that there is a set of places v of k of positive density such that $Q_v \notin NA(k_v)$. Excluding the finitely many places dividing $N\infty$ or of bad reduction for A does not change this density. For v in this reduced set, we have $A(k_v)/NA(k_v) \cong A(\mathbb{F}_v)/NA(\mathbb{F}_v)$, and so the image of $\hat{Q}$ in $A(\mathbb{F}_v)$ is not in $NA(\mathbb{F}_v)$, let alone zero.

Now consider the case that $\hat{Q} \in A(k)_{\text{tors}} \setminus \{0\}$. We know that for all but finitely many finite places v of good reduction, $A(k)_{\text{tors}}$ injects into $A(\mathbb{F}_v)$, so in this case, the statement is even true for a set of places of density 1. □

Remark 3.9. Note that the corresponding statement for points $Q \in A(k)$ is trivial; indeed, there are only finitely many finite places v of good reduction such that Q maps trivially into $A(\mathbb{F}_v)$. (Consider some projective model of A; then Q and 0 are two distinct points in projective space. They will reduce to the same point mod v if and only if v divides certain nonzero numbers (2×2 determinants formed with the coordinates of the two points).) The lemma above says that things can not go wrong too badly when we replace $A(k)$ by its completion $\widehat{A(k)}$ or even $\widehat{\operatorname{Sel}}(k, A)$.

Theorem 3.10. *Let S be a set of finite places of k of good reduction for A and of density 1. Then the canonical homomorphisms*

$$\widehat{\operatorname{Sel}}(k, A) \longrightarrow \prod_{v \in S} A(\mathbb{F}_v) \quad \textit{and} \quad \widehat{A(k)} \longrightarrow \prod_{v \in S} A(\mathbb{F}_v)$$

are injective.

Proof: Let $\hat{Q}$ be in the kernel. If $\hat{Q} \neq 0$, then by Lemma 3.8, there is a set of places v of positive density such that the image of $\hat{Q}$ in $A(\mathbb{F}_v)$ is nonzero, contradicting the assumptions. So $\hat{Q} = 0$, and the map is injective. □

For applications, it is useful to remove the condition in Prop. 3.6 that all points of Z have to be defined over k.

Theorem 3.11. *Let $Z \subset A$ be a finite subscheme of an abelian variety A over k. Let $\hat{P} \in \widehat{\operatorname{Sel}}(k, A)$ be such that $P_v \in Z(k_v)$ for a set of places v of k of density 1. Then $\hat{P}$ is in the image of $Z(k)$ in $\widehat{\operatorname{Sel}}(k, A)$.*

Proof: Let K/k be a finite extension such that $Z(K) = Z(\bar{k})$. By Prop. 3.6, we have that the image of $\hat{P}$ in $A(\mathbb{A}_K)_\bullet$ is in $Z(K)$. This implies that the image of $\hat{P}$ in $A(\mathbb{A}_K)_\bullet$ is in $Z(k)$ (since $\hat{P}$ is k-rational). Now the canonical map $A(\mathbb{A}_k)_\bullet \to A(\mathbb{A}_K)_\bullet$ is injective except possibly at some of the infinite places, so $P_v \in Z(k)$ for all but finitely many places. Now, replacing Z by $Z(k)$ and applying Prop. 3.6 again (this time over k), we find that $\hat{P} \in Z(k)$, as claimed. □

We have seen that for zero-dimensional subvarieties $Z \subset A$, we have $Z(\mathbb{A}_k)_\bullet \cap \overline{A(k)} = Z(k)$, or even more generally, $Z(\mathbb{A}_k)_\bullet \cap \widehat{\operatorname{Sel}}(k, A) = Z(k)$ (writing intersections for simplicity). One can ask if this is valid more generally for subvarieties $X \subset A$ that do not contain the translate of an abelian subvariety of positive dimension.

Question 3.12. Is there such a thing as an "Adelic Mordell-Lang Conjecture"?

A possible statment is as follows. Let A/k be an abelian variety and $X \subset A$ a subvariety not containing the translate of a nontrivial subabelian variety of A. Then there is a finite subscheme $Z \subset X$ such that

$$X(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(k, A) \subset Z(\mathbb{A}_k)_\bullet \,.$$

If this holds, Thm. 3.11 above implies that

$$X(k) \subset X(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(k, A) \subset Z(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(k, A) = Z(k) \subset X(k)$$

and therefore $X(k) = X(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(k, A)$. In the notation introduced in Section 5 below and by the discussion in Section 6, this implies

$$X(k) \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet \cap A(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(k, A) = X(k)\,,$$

and so X is excellent w.r.t. abelian coverings (and hence "very good").

Remark 3.13. Note that the Adelic Mordell Lang Conjecture formulated above is true when k is a global function field, A is ordinary, and X is not defined over k^p (where p is the characteristic of k), see Voloch's paper [Vo]. (The result is also implicit in [Hr].)

4. Torsors and twists

In this section, we introduce the notions of torsors (under finite étale group schemes) and twists, and we describe various constructions that can be done with these objects.

Let X be a smooth projective (reduced, but not necessarily geometrically connected) variety over k.

We will consider the following category $\mathcal{C}ov(X)$. Its objects are X-torsors Y under G (see for example [Sk2] for definitions), where G is a finite étale group scheme over k. More concretely, the data consists of a k-morphism $\mu : Y \times G \to Y$ describing an action of G on Y, together with a finite étale k-morphism $\pi : Y \to X$ such that the following diagram is cartesian (i.e., identifies $Y \times G$ with the fiber product $Y \times_X Y$).

$$\begin{array}{ccc} Y \times G & \xrightarrow{\mu} & Y \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\pi} & X \end{array}$$

We will usually just write (Y, G) for such an object, with the maps μ and π being understood. Morphisms $(Y', G') \to (Y, G)$ in $\mathcal{C}ov(X)$ are given by a pair of maps

(k-morphisms of (group) schemes) $\phi : Y' \to Y$ and $\gamma : G' \to G$ such that the obvious diagram commutes:

$$\begin{array}{ccccc} Y' \times G' & \xrightarrow{\mu'} & Y' & \xrightarrow{\pi'} & X \\ \downarrow{\scriptstyle \phi\times\gamma} & & \downarrow{\scriptstyle \phi} & & \| \\ Y \times G & \xrightarrow{\mu} & Y & \xrightarrow{\pi} & X \end{array}$$

We will denote by $\mathcal{S}ol(X)$ and $\mathcal{A}b(X)$ the full subcategories of $\mathcal{C}ov(X)$ whose objects are the torsors (Y, G) such that G is solvable or abelian, respectively.

If $X' \to X$ is a k-morphism of (smooth projective) varieties, then we can pull back X-torsors under G to obtain X'-torsors under G. This defines covariant functors $\mathcal{C}ov(X) \to \mathcal{C}ov(X')$, $\mathcal{S}ol(X) \to \mathcal{S}ol(X')$ and $\mathcal{A}b(X) \to \mathcal{A}b(X')$.

The following constructions are described for $\mathcal{C}ov(X)$, but they are similarly valid for $\mathcal{S}ol(X)$ and $\mathcal{A}b(X)$.

If $(Y_1, G_1), (Y_2, G_2) \in \mathcal{C}ov(X)$ are two X-torsors, then we can construct their fiber product $(Y, G) \in \mathcal{C}ov(X)$, where $Y = Y_1 \times_X Y_2$ and $G = G_1 \times G_2$. More generally, If $(Y_1, G_1) \to (Y, G)$ and $(Y_2, G_2) \to (Y, G)$ are two morphisms in $\mathcal{C}ov(X)$, there is a fiber product $(Z, H) \in \mathcal{C}ov(X)$, where $Z = Y_1 \times_Y Y_2$ and $H = G_1 \times_G G_2$.

If $(Y, G) \in \mathcal{C}ov(X)$ is an X-torsor, where now everything is over K with a finite extension K/k, then we can apply restriction of scalars to obtain $(R_{K/k}Y, R_{K/k}G) \in \mathcal{C}ov(R_{K/k}X)$.

If $(Y, G) \in \mathcal{C}ov(X)$ is an X-torsor and ξ is a cohomology class in $H^1(k, G)$, then we can construct the *twist* (Y_ξ, G_ξ) of (Y, G) by ξ. Here G_ξ is the inner form of G corresponding to ξ (compare, e.g., [Sk2, pp. 12, 20]). We will denote the structure maps by μ_ξ and π_ξ. Usually, $H^1(k, G)$ is just a pointed set with distinguished element corresponding to the given torsor; if the torsor is abelian, $H^1(k, G)$ is a group, and $G_\xi = G$ for all $\xi \in H^1(k, G)$.

If $(\phi, \gamma) : (Y', G') \to (Y, G)$ is a morphism and $\xi \in H^1(k, G')$, then we get an induced morphism $(Y'_\xi, G'_\xi) \to (Y_{\gamma_*\xi}, G_{\gamma_*\xi})$ (where γ_* is the induced map $H^1(k, G') \to H^1(k, G)$). Similarly, twists are compatible with pull-backs, fiber products and restriction of scalars.

Twists are transitive in the following sense. If $(Y, G) \in \mathcal{C}ov(X)$ is an X-torsor and $\xi \in H^1(k, G)$, $\eta \in H^1(k, G_\xi)$, then there is a $\zeta \in H^1(k, G)$ such that $((Y_\xi)_\eta, (G_\xi)_\eta) \cong (Y_\zeta, G_\zeta)$. Conversely, if ξ and ζ are given, then there is an $\eta \in H^1(k, G_\xi)$ such that the relation above holds.

The following observation does not hold in general for $\mathcal{S}ol(X)$ and $\mathcal{A}b(X)$. If $Y \xrightarrow{\pi} X$ is any finite étale morphism, then there is some $(\tilde{Y}, G) \in \mathcal{C}ov(X)$ such that $\tilde{\pi} : \tilde{Y} \to X$ factors through π. Also, if we have $(Y, G) \in \mathcal{C}ov(X)$ and $(Z, H) \in \mathcal{C}ov(Y)$, then there is some $(\tilde{Z}, \Gamma) \in \mathcal{C}ov(X)$ such that $\tilde{Z}$ maps to Z over X and such that the induced map $\tilde{Z} \to Y$ gives rise to a Y-torsor $(\tilde{Z}, \tilde{H}) \in \mathcal{C}ov(Y)$. This last statement is also valid with $\mathcal{S}ol(X)$ and $\mathcal{S}ol(Y)$ in place of $\mathcal{C}ov(X)$ and $\mathcal{C}ov(Y)$ (since extensions of solvable groups are solvable).

5. Finite descent conditions

In this section, we use torsors and their twists, as described in the previous section, in order to obtain obstructions against rational points. The use of torsors under finite abelian group schemes is classical; it is what is behind the usual descent procedures on elliptic curves or abelian varieties (and so one can claim that they go all the way back to Fermat). The non-abelian case was first studied by Harari and Skorobogatov [HS1]; see also [Ha2].

The following theorem (going back to Chevalley and Weil [CW]) summarizes the standard facts about descent via torsors. Compare also [HS1, Lemma 4.1] and [Sk2, pp. 105, 106].

Theorem 5.1. *Let $(Y, G) \in \mathcal{C}ov(X)$ be a torsor.*

(1) $X(k) = \coprod_{\xi \in H^1(k,G)} \pi_\xi(Y_\xi(k))$.

(2) *The (Y, G)-*Selmer set

$$\mathrm{Sel}^{(Y,G)}(k, X) = \{\xi \in H^1(k, G) : Y_\xi(\mathbb{A}_k)_\bullet \neq \emptyset\}$$

is finite: there are only finitely many twists (Y_ξ, G_ξ) such that Y_ξ has points everywhere locally.

At least in principle, the Selmer set in the second statement can be determined explicitly, and the union in the first statement can be restricted to this finite set.

The idea behind the following considerations is to see how much information one can get out of the various torsors regarding the image of $X(k)$ in $X(\mathbb{A}_k)_\bullet$. Compare Definition 4.2 in [HS1] and Definition 5.3.1 in Skorobogatov's book [Sk2].

Definition 5.2. Let $(Y, G) \in \mathcal{C}ov(X)$ be an X-torsor. We say that a point $P \in X(\mathbb{A}_k)_\bullet$ *survives* (Y, G), if it lifts to a point in $Y_\xi(\mathbb{A}_k)_\bullet$ for some twist (Y_ξ, G_ξ) of (Y, G).

There is a cohomological description of this property. An X-torsor under G is given by an element of $H^1_{\text{ét}}(X, G)$. Pull-back through the map $\operatorname{Spec} k \to X$ corresponding to a point in $X(k)$ gives a map

$$X(k) \longrightarrow H^1(k, G)\,.$$

Note that it is not necessary to refer to non-abelian étale cohomology here: the map $X(k) \to H^1(k, G)$ induced by a torsor (Y, G) simply arises by associating to a point $P \in X(k)$ its fiber $\pi^{-1}(P) \subset Y$, which is a k-torsor under G and therefore corresponds to an element of $H^1(k, G)$.

We get a similar map on adelic points:

$$X(\mathbb{A}_k)_\bullet \longrightarrow \prod_v H^1(k_v, G)$$

There is the canonical restriction map

$$H^1(k, G) \longrightarrow \prod_v H^1(k_v, G)\,,$$

and the various maps piece together to give a commutative diagram:

$$\begin{array}{ccc} X(k) & \longrightarrow & H^1(k, G) \\ \big\downarrow & & \big\downarrow \\ X(\mathbb{A}_k)_\bullet & \longrightarrow & \prod_v H^1(k_v, G) \end{array}$$

A point $P \in X(\mathbb{A}_k)_\bullet$ survives (Y, G) if and only if its image in $\prod_v H^1(k_v, G)$ is in the image of the global set $H^1(k, G)$. The (Y, G)-Selmer set is then the preimage in $H^1(k, G)$ of the image of $X(\mathbb{A}_k)_\bullet$; this is completely analogous to the definition of a Selmer group in case X is an abelian variety A, and $G = A[n]$ is the n-torsion subgroup of A.

Here are some basic properties.

Lemma 5.3.

(1) *If $(\phi, \gamma) : (Y', G') \to (Y, G)$ is a morphism in $\mathcal{C}ov(X)$, and if $P \in X(\mathbb{A}_k)_\bullet$ survives (Y', G'), then P also survives (Y, G).*

(2) *If $(Y', G) \in \mathcal{C}ov(X')$ is the pull-back of $(Y, G) \in \mathcal{C}ov(X)$ under a morphism $\psi : X' \to X$, then $P \in X'(\mathbb{A}_k)_\bullet$ survives (Y', G) if and only if $\psi(P)$ survives (Y, G).*

(3) *If $(Y_1, G_1), (Y_2, G_2) \in \mathcal{C}ov(X)$ have fiber product (Y, G), then $P \in X(\mathbb{A}_k)_\bullet$ survives (Y, G) if and only if P survives both (Y_1, G_1) and (Y_2, G_2).*

(4) *Let X be over K, where K/k is a finite extension, and let $(Y, G) \in \mathcal{C}ov(X)$ be an X-torsor. Then $P \in (R_{K/k}X)(\mathbb{A}_k)_\bullet$ survives $(R_{K/k}Y, R_{K/k}G)$, if and only if its image in $X(\mathbb{A}_K)_\bullet$ survives (Y, G).*

(5) *If $(Y, G) \in \mathcal{C}ov(X)$ and $\xi \in H^1(k, G)$, then $P \in X(\mathbb{A}_k)_\bullet$ survives (Y, G) if and only if P survives (Y_ξ, G_ξ).*

Proof:

(1) By assumption, there are $\xi \in H^1(k, G')$ and $Q \in Y'_\xi(\mathbb{A}_k)_\bullet$ such that $\pi'_\xi(Q) = P$. Now we have the morphism $\phi_\xi : Y'_\xi \to Y_{\gamma_*\xi}$ over X, hence $\pi_{\gamma_*\xi}(\phi_\xi(Q)) = \pi'_\xi(Q) = P$, whence P survives (Y, G).

(2) Assume that P survives (Y', G). Then there are $\xi \in H^1(k, G)$ and $Q \in Y'_\xi(\mathbb{A}_k)_\bullet$ such that $\pi'_\xi(Q) = P$. There is a morphism $\Psi_\xi : Y'_\xi \to Y_\xi$ over ψ, hence we have that $\pi_\xi(\Psi_\xi(Q)) = \psi(P)$, so $\psi(P)$ survives (Y, G). Conversely, assume that $\psi(P)$ survives (Y, G). Then there are $\xi \in H^1(k, G)$ and $Q \in Y_\xi(\mathbb{A}_k)_\bullet$ such that $\pi_\xi(Q) = \psi(P)$. The twist (Y'_ξ, G_ξ) is the pull-back of (Y_ξ, G_ξ) under ψ; in particular, $Y'_\xi = Y_\xi \times_X X'$, and so there is $Q' \in Y'_\xi(\mathbb{A}_k)_\bullet$ mapping to Q in Y_ξ and to P in X'. Hence P survives (Y', G).

(3) We have obvious morphisms $(Y, G) \to (Y_i, G_i)$. So by part (1), if P survives (Y, G), then it also survives (Y_1, G_1) and (Y_2, G_2). Now assume that P survives both (Y_1, G_1) and (Y_2, G_2). Then there are $\xi_1 \in H^1(k, G_1)$ and $\xi_2 \in H^1(k, G_2)$ and points $Q_1 \in Y_{1,\xi_1}(\mathbb{A}_k)_\bullet$, $Q_2 \in Y_{2,\xi_2}(\mathbb{A}_k)_\bullet$ such that $\pi_{1,\xi_1}(Q_1) = P$ and $\pi_{2,\xi_2}(Q_2) = P$. Consider $\xi = (\xi_1, \xi_2) \in H^1(k, G) = H^1(k, G_1) \times H^1(k, G_2)$. We have that $Y_\xi = Y_{1,\xi_1} \times_X Y_{2,\xi_2}$, hence there is $Q \in Y_\xi(\mathbb{A}_k)_\bullet$ mapping to Q_1 and Q_2 under the canonical maps $Y_\xi \to Y_{i,\xi_i}$ $(i = 1, 2)$, and to P under $\pi_\xi : Y_\xi \to X$. Hence P survives (Y, G).

(4) We have $H^1(k, R_{K/k}G) = H^1(K, G)$, and the corresponding twists are compatible. For any ξ in this set, we have $R_{K/k}Y_\xi = (R_{K/k}Y)_\xi$, and the adelic points $(R_{K/k}Y_\xi)(\mathbb{A}_k)_\bullet$ and $Y_\xi(\mathbb{A}_K)_\bullet$ are identified. The claim follows.

(5) This comes from the fact that every twist of (Y, G) is also a twist of (Y_ξ, G_ξ) and vice versa.

□

By the Descent Theorem 5.1, it is clear that (the image in $X(\mathbb{A}_k)_\bullet$ of) a rational point $P \in X(k)$ survives ervery torsor. Therefore it makes sense to study the set of adelic points that survive every torsor (or a suitable subclass of torsors) in order to obtain information on the location of the rational points within the adelic points. Note that the set of points in $X(\mathbb{A}_k)_\bullet$ surviving a given torsor is closed — it is a finite union of images of compact sets $Y_\xi(\mathbb{A}_k)_\bullet$ under continuous maps.

We are led to the following definitions.

Definition 5.4. Let X be a smooth projective variety over k.

(1) $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \{P \in X(\mathbb{A}_k)_\bullet : P \text{ survives all } (Y, G) \in \mathcal{C}ov(X)\}$.
(2) $X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = \{P \in X(\mathbb{A}_k)_\bullet : P \text{ survives all } (Y, G) \in \mathcal{S}ol(X)\}$.
(3) $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \{P \in X(\mathbb{A}_k)_\bullet : P \text{ survives all } (Y, G) \in \mathcal{A}b(X)\}$.

(The "f" in the superscripts stands for "finite", since we are dealing with torsors under finite group schemes only.)

By the remark made before the definition above, we have

$$X(k) \subset \overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-sol}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet\,.$$

Here, $\overline{X(k)}$ is the topological closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet$.

Recall the "evaluation map" for $P \in X(\mathbb{A}_k)_\bullet$ and G a finite étale k-group scheme,

$$\mathrm{ev}_{P,G} : H^1_{\text{ét}}(X, G) \longrightarrow \prod_v H^1(k_v, G)$$

(the set on the left can be considered as the set of isomorphism classes of X-torsors under G) and the restriction map

$$\mathrm{res}_G : H^1(k, G) \longrightarrow \prod_v H^1(k_v, G)\,.$$

In these terms, we have

$$X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \bigcap_G \{P \in X(\mathbb{A}_k)_\bullet : \mathrm{im}(\mathrm{ev}_{P,G}) \subset \mathrm{im}(\mathrm{res}_G)\}\,,$$

where G runs through all finite étale k-group schemes. We obtain $X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$ and $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ in a similar way, by restricting G to solvable or abelian group schemes.

In the definition above, we can restrict to (Y, G) with Y connected (over k) if X is connected: if we have (Y, G) with Y not connected, then let Y_0 by a connected component of Y, and let $G_0 \subset G$ be the stabilizer of this component. Then (Y_0, G_0) is again a torsor of the same kind as (Y, G), and we have a morphism $(Y_0, G_0) \to (Y, G)$. Hence, by Lemma 5.3, (1), if P survives (Y_0, G_0), then it also survives (Y, G).

However, we cannot restrict to geometrically connected torsors when X is geometrically connected. The reason is that there can be obstructions coming from the fact that a suitable geometrically connected torsor does not exist.

Lemma 5.5. *Assume that X is geometrically connected. If there is a torsor $(Y, G) \in \mathcal{C}ov(X)$ such that Y and all twists Y_ξ are k-connected, but not geometrically connected, then $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \emptyset$. The analogous statement holds for the solvable and abelian versions.*

Proof: If Y_ξ is connected, but not geometrically connected, then $Y_\xi(\mathbb{A}_k)_\bullet = \emptyset$ (this is because the finite scheme $\pi_0(Y_\xi)$ is irreducible and therefore satisfies the Hasse Principle, compare the proof of Prop. 5.17). Hence no point in $X(\mathbb{A}_k)_\bullet$ survives (Y, G). □

Let us briefly discuss how this relates to the geometric fundamental group of X over $\bar{k}$, assuming X to be geometrically connected. In the following, we write

$\bar{X} = X \times_k \bar{k}$ etc., for the base-change of X to a variety over $\bar{k}$. Every torsor $(Y, G) \in \mathcal{C}ov(X)$ ($\mathcal{S}ol(X)/\mathcal{A}b(X)$) gives rise to a covering $\bar{Y} \to \bar{X}$ that is Galois with (solvable/abelian) Galois group $G(\bar{k})$. The stabilizer Γ of a connected component of $\bar{Y}$ is then a finite quotient of the geometric fundamental group $\pi_1(\bar{X})$. If we fix an embedding $k \to \mathbb{C}$, then $\pi_1(\bar{X})$ is the pro-finite completion of the topological fundamental group $\pi_1(X(\mathbb{C}))$, so Γ is also a finite quotient of $\pi_1(X(\mathbb{C}))$. If Γ is trivial, then $\pi_0(Y)$ is a k-torsor under G, and (Y, G) is the pull-back of $(\pi_0(Y), G)$ under the structure morphism $X \to \operatorname{Spec} k$. We call such a torsor *trivial*. Note that all points in $X(\mathbb{A}_k)_\bullet$ survive a trivial torsor (since their image in $(\operatorname{Spec} k)(\mathbb{A}_k)_\bullet = (\operatorname{Spec} k)(k) = \{\text{pt}\}$ survives everything); therefore trivial torsors do not give information.

Conversely, given a finite quotient Γ of $\pi_1(\bar{X})$ or of $\pi_1(X(\mathbb{C}))$, there is a corresponding covering $\bar{Y} \to \bar{X}$ that will be defined over some finite extension K of k. Let $\pi : Y \to X_K$ be the covering over K; it is a torsor under a K-group scheme G such that $G(\bar{k}) = \Gamma$. We now construct a torsor $(Z, R_{K/k}G) \in \mathcal{C}ov(X)$ that over K factors through π. By restriction of scalars, we obtain $(R_{K/k}Y, R_{K/k}G) \in \mathcal{C}ov(R_{K/k}X_K)$. We pull back via the canoncial morphism $X \to R_{K/k}X_K$ to obtain $(Z, R_{K/k}G) \in \mathcal{C}ov(X)$. Over K, we have the following diagram.

$$\begin{array}{ccccc} Z_K & \longrightarrow & (R_{K/k}Y)_K & \xrightarrow{\text{can}} & Y \\ \big\downarrow {\scriptstyle (R_{K/k}G)_K} & & \big\downarrow {\scriptstyle (R_{K/k}G)_K} & & \big\downarrow {\scriptstyle G} \\ X_K & \xrightarrow{\text{can}} & (R_{K/k}X_K)_K & \xrightarrow{\text{can}} & X_K \end{array}$$

The composition of the lower horizontal maps is the identity morphism, hence $(Z_K, (R_{K/k}G)_K) \in \mathcal{C}ov(X_K)$ maps to (Y, G). Note that the torsor we construct is in $\mathcal{S}ol(X)$ ($\mathcal{A}b(X)$) when Γ is solvable (abelian).

Lemma 5.6. *Let X be geometrically connected.*

(1) *If $\pi_1(\bar{X})$ is trivial (i.e., X is simply connected), then $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet$.*
(2) *If the abelianization $\pi_1(\bar{X})^{\text{ab}}$ is trivial, then $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet$.*
(3) *If $\pi_1(\bar{X})$ is abelian and $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \neq \emptyset$, then $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$.*

PROOF:

(1) In this case, all torsors are trivial and are therefore survived by all points in $X(\mathbb{A}_k)_\bullet$.
(2) Here the same holds for all abelian torsors.
(3) We always have $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. So let $P \in X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, and let $(Y, G) \in \mathcal{C}ov(X)$ be a torsor. Since $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \neq \emptyset$, by Lemma 5.5 there must be some twist (Y_ξ, G_ξ) of (Y, G) such that a geometric component Y_0 of Y_ξ is defined over k. Let Γ be the stabilizer of Y_0 in G_ξ; then (Y_0, Γ) is a

torsor mapping to (Y_ξ, G_ξ). Since Y_0 is geometrically connected, $\Gamma(\bar{k})$ is a quotient of the fundmental group $\pi_1(X)$ and hence abelian. Therefore P survives (Y_0, Γ) and hence also (Y_ξ, G_ξ), or equivalently, (Y, G).

□

Remark 5.7. In statement (3) above, it seems hardly conceivable that $X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ could be empty, while $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ is not.

We now list some fairly elementary properties of the sets $X(\mathbb{A}_k)_\bullet^{\text{f-ab/f-sol/f-cov}}$.

Proposition 5.8. *If $X' \xrightarrow{\psi} X$ is a morphism, then $\psi(X'(\mathbb{A}_k)_\bullet^{\text{f-cov}}) \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$. Similarly for the solvable and abelian variants.*

PROOF: Let $P \in X'(\mathbb{A}_k)_\bullet^{\text{f-cov}}$, and let $(Y, G) \in \mathcal{C}ov(X)$ be an X-torsor. By assumption, P survives the pull-back (Y', G) of (Y, G) under ψ, so by Lemma 5.3, part (2), $\psi(P)$ survives (Y, G). Since (Y, G) was arbitrary, $\psi(P) \in X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$. The same proof works for the solvable and abelian variants. □

Remark 5.9. The preceding result implies that

$$(X \times Y)(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \times Y(\mathbb{A}_k)_\bullet^{\text{f-cov}}$$

(and similarly for the solvable and abelian variants). One would actually expect equality here. This is in fact true, see Prop. 9.14 below.

Proposition 5.10. *If K/k is a finite extension and X is a K-variety, then*

$$(R_{K/k}X)(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_K)_\bullet^{\text{f-cov}}$$

(under the canonical identification $(R_{K/k}X)(\mathbb{A}_k)_\bullet = X(\mathbb{A}_K)_\bullet$), and similarly for the solvable and abelian variants.

PROOF: Let $P \in (R_{K/k}X)(\mathbb{A}_k)_\bullet^{\text{f-cov}}$, and let $(Y, G) \in \mathcal{C}ov(X)$. By assumption, P survives $(R_{K/k}Y, R_{K/k}G) \in \mathcal{C}ov(R_{K/k}X)$, so by Lemma 5.3, part (4), P also survives (Y, G). Since (Y, G) was arbitrary, $P \in X(\mathbb{A}_K)_\bullet^{\text{f-cov}}$. The same proof works for the solvable and abelian variants. □

Remark 5.11. One would expect to actually have equality. To prove this, one would have to show that for every $(Y', G') \in \mathcal{C}ov(R_{K/k}X)$, there is a $(Y, G) \in \mathcal{C}ov(X)$ such that $(R_{K/k}Y, R_{K/k}G)$ maps to (Y', G').

Proposition 5.12. *If K/k is a finite extension, then*

$$X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet \cap X(\mathbb{A}_K)_\bullet^{\text{f-cov}}$$

and similarly for the solvable and abelian variants. Note that the intersection is to be interpreted as the pullback of $X(\mathbb{A}_K)_\bullet^{\text{f-cov}}$ under the canonical map $X(\mathbb{A}_k)_\bullet \to X(\mathbb{A}_K)_\bullet$, which may not be injective at the infinite places.

Proof: We have a morphism $X \to R_{K/k}X_K$, inducing the canonical map

$$X(\mathbb{A}_k)_\bullet \longrightarrow (R_{K/k}X_K)(\mathbb{A}_k)_\bullet = X(\mathbb{A}_K)_\bullet\,.$$

The claim now follows from combining Props. 5.8 and 5.10. □

We also have an analogue of the Descent Theorem 5.1.

Proposition 5.13. *Let $(Y,G) \in \mathcal{C}ov(X)$ be an X-torsor. Then*

$$X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \bigcup \pi_\xi\bigl(Y_\xi(\mathbb{A}_k)_\bullet^{\text{f-cov}}\bigr)\,,$$

where the union is extended over all twists (Y_ξ, G_ξ) of (Y,G), or equivalently, over the finite set of twists with points everywhere locally. A similar statement holds for the solvable variant, when G is solvable.

Proof: Note first that by Prop. 5.8, the right hand side is a subset of the left hand side.

For the reverse inclusion, take $P \in X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$. To ease notation, we will suppress the group schemes when denoting torsors in the following. Let $Y_1, \dots, Y_s \in \mathcal{C}ov(X)$ (or $\mathcal{S}ol(X)$) be the finitely many twists of Y such that P lifts.

Define $\tau(j) \subset \{1, \dots, s\}$ to be the set of indices i such that for every X-torsor Z mapping to Y_j (or short: an X-torsor Z over Y_j), there is a twist Z_ξ that lifts P and induces a twist of Y_j that is isomorphic to Y_i. We make a number of claims about this function.

(i) $\tau(j)$ is non-empty.

To see this, note first that for any given Z, the corresponding set (call it $\tau(Z)$) is non-empty, since by assumption P must lift to some twist of Z, and this twist induces a twist of Y_j to which P also lifts, hence this twist must be one of the Y_i. Second, if Z maps to Z' (as X-torsors over Y_j), we have $\tau(Z) \subset \tau(Z')$. Third, for every pair of X-torsors Z and Z' over Y_j, their relative fiber product $Z \times_{Y_j} Z'$ maps to both of them. Taking these together, we see that $\tau(j)$ is a filtered intersection of non-empty subsets of a finite set and hence non-empty.

(ii) If $i \in \tau(j)$, then $\tau(i) \subset \tau(j)$.

Let $h \in \tau(i)$, and let Z be an X-torsor over Y_j. By definition of $\tau(j)$, there is a twist Z_ξ of Z lifting P and inducing the twist Y_i of Y_j. Now by definition of $\tau(i)$, there is a twist $(Z_\xi)_\eta$ of Z_ξ lifting P and inducing the twist Y_h of Y_i. By transitivity of twists, this means that we have a twist of Z lifting P and inducing the twist Y_h of Y_j. Since Z was arbitrary, this shows that $h \in \tau(j)$.

(iii) For some j, we have $j \in \tau(j)$.

Indeed, selecting for each j some $\sigma(j) \in \tau(j)$ (this is possible by (i)), the map σ will have a cycle: $\sigma^m(j) = j$ for some $m \geq 1$ and j. Then by (ii), it follows that $j \in \tau(j)$.

For this specific value of j, we have therefore proved that every X-torsor Z over Y_j has a twist that lifts P and induces the trivial twist of Y_j. This means in particular that this twist is also a twist of Z as a Y_j-torsor.

Now assume that P does not lift to $Y_j(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ (or $Y_j(\mathbb{A}_k)_\bullet^{\text{f-sol}}$). Since the preimages of P in $Y_j(\mathbb{A}_k)_\bullet$ form a compact set and since surviving a torsor is a closed condition, we can find a Y_j-torsor V that is not survived by any of the preimages of P. We can then find an X-torsor Z mapping to V, staying in $\mathcal{S}ol$ when working in that category. (Note that this step does not work for $\mathcal{A}b$, since extensions of abelian groups need not be abelian again.) But by what we have just proved, Z has a twist as a Y_j-torsor that lifts a preimage of P, a contradiction. Hence our assumption that P does not lift to $Y_j(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ (or $Y_j(\mathbb{A}_k)_\bullet^{\text{f-sol}}$) must be false. □

Remark 5.14. The analogous statement for $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ and G abelian is not true in general: it would follow that $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$, but Skorobogatov (see [Sk2, § 8] or [Sk1]) has a celebrated example of a surface X such that $\emptyset = X(\mathbb{A}_k)_\bullet^{\text{f-sol}} \subsetneq X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. In fact, there is an abelian covering $\pi : Y \to X$ such that $\bigcup_\xi \pi_\xi(Y_\xi(\mathbb{A}_k)_\bullet^{\text{f-ab}}) = \emptyset$, which therefore gives a counterexample to the abelian version of the statement.

Skorobogatov shows that the "Brauer set" $X(\mathbb{A}_k)_\bullet^{\text{Br}}$ is nonempty. In a later paper [HS1, § 5.1], Harari and Skorobogatov show that there exists an obstruction coming from a nilpotent, non-abelian covering (arising from an abelian covering of Y). The latter means that $X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = \emptyset$, whereas the former implies that $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \neq \emptyset$, since $X(\mathbb{A}_k)_\bullet^{\text{Br}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$; see Section 7 below. The interest in this result comes from the fact that it is the first example known of a variety where there is no Brauer-Manin obstruction, yet there are no rational points.

Lemma 5.15. *Let* $Z = \operatorname{Spec} k \amalg \operatorname{Spec} k = \{P_1, P_2\}$. *Then*

$$\{P_1, P_2\} = Z(k) = Z(\mathbb{A}_k)_\bullet^{\text{f-ab}}.$$

Proof: Let $Q \in Z(\mathbb{A}_k)_\bullet$ and assume that $Q \notin Z(k)$. We have to show that $Q \notin Z(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. By assumption, there are places v and w of k such that $Q_v = P_1$ and $Q_w = P_2$. We will consider torsors under $G = \mathbb{Z}/2\mathbb{Z}$. Pick some $\alpha \in k^\times$ such that $\alpha \notin (k_v^\times)^2$ and $\alpha \notin (k_w^\times)^2$. Let $Y = \operatorname{Spec} k(\sqrt{\alpha}) \amalg (\operatorname{Spec} k \amalg \operatorname{Spec} k)$; then $(Y, G) \in \mathcal{A}b(Z)$ in an obvious way. We want to show that no twist (Y_ξ, G) for $\xi \in H^1(k, G) = k^\times/(k^\times)^2$ lifts Q. Such a twist is of one of the following forms.

$$(Y_\xi, G) = \operatorname{Spec} k(\sqrt{\alpha}) \amalg (\operatorname{Spec} k \amalg \operatorname{Spec} k)$$
$$(Y_\xi, G) = (\operatorname{Spec} k \amalg \operatorname{Spec} k) \amalg \operatorname{Spec} k(\sqrt{\alpha})$$
$$(Y_\xi, G) = \operatorname{Spec} k(\sqrt{\beta}) \amalg \operatorname{Spec} k(\sqrt{\gamma})$$

where in the last case, β and γ are independent in $k^\times/(k^\times)^2$. In the first two cases, Q does not lift, since in the first case, the first component does not lift Q_v, and in the second case, the second component does not lift Q_w (by our choice of α). In the third case, there is a set of places of k of density $1/4$ that are inert in both $k(\sqrt{\beta})$ and $k(\sqrt{\gamma})$, so that $Y_\xi(\mathbb{A}_k)_\bullet = \emptyset$. In particular, Q does not lift to any of these twists. □

Proposition 5.16. *If $X = X_1 \amalg X_2 \amalg \cdots \amalg X_n$ is a disjoint union, then*

$$X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \coprod_{j=1}^{n} X_j(\mathbb{A}_k)_\bullet^{\text{f-cov}} ,$$

and similarly for the solvable and abelian variants.

PROOF: It is sufficient to consider the case $n = 2$. We have maps $X_1 \to X$ and $X_2 \to X$, so (by Prop. 5.8) $X_1(\mathbb{A}_k)_\bullet^{\text{f-cov}} \amalg X_2(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ (same for $\cdot^{\text{f-sol}}$ and $\cdot^{\text{f-ab}}$). For the reverse inclusion, consider the morphism $X \to \operatorname{Spec} k \amalg \operatorname{Spec} k = Z$ mapping X_1 to the first point and X_2 to the second point. If $Q \in X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, then its image is in $Z(\mathbb{A}_k)_\bullet^{\text{f-ab}} = Z(k)$ (by Prop. 5.8 again and Lemma 5.15). This means that $Q \in X_1(\mathbb{A}_k)_\bullet \amalg X_2(\mathbb{A}_k)_\bullet$. The claim then follows easily. □

Proposition 5.17. *If Z is a (reduced) finite scheme, then $Z(\mathbb{A}_k)_\bullet^{\text{f-ab}} = Z(k)$.*

PROOF: By Prop. 5.16, it suffices to prove this when $Z = \operatorname{Spec} K$ is connected. But in this case, it is known that Z satisfies the Hasse Principle. On the other hand, if $Z(k) \neq \emptyset$, then $Z = \operatorname{Spec} k$ and $Z(\mathbb{A}_k)_\bullet$ has just one point, so $Z(k) = Z(\mathbb{A}_k)_\bullet$.

(The statement that $\operatorname{Spec} K$ as a k-scheme satisfies the Hasse Principle comes down to the following fact:

If a group G acts transitively on a finite set X such that every $g \in G$ fixes at least one element of X, then $\#X = 1$.

To see this, let $n = \#X$ and assume (w.l.o.g.) that $G \subset S_n$. The stabilizer G_x of $x \in X$ is a subgroup of index n in G. By assumption, $G = \bigcup_{x \in X} G_x$, so $G \setminus \{1\} = \bigcup_{x \in X} (G_x \setminus \{1\})$. Counting elements now gives $\#G - 1 \le n(\#G/n - 1) = \#G - n$, which implies $n = 1$.) □

Remark 5.18. Note that the Hasse Principle does not hold in general for finite schemes. A typical counterexample is given by the $\mathbb{Q}$-scheme

$$\operatorname{Spec} \mathbb{Q}(\sqrt{13}) \amalg \operatorname{Spec} \mathbb{Q}(\sqrt{17}) \amalg \operatorname{Spec} \mathbb{Q}(\sqrt{13 \cdot 17}) .$$

6. Finite descent conditions and rational points

The ultimate goal behind considering the sets cut out in the adelic points by the various covering conditions is to obtain information on the rational points.

There is a three-by-three matrix of natural statements relating these sets, see the diagram below. Here, $\overline{X(k)}$ is the topological closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet$.

(6.1)

$$\begin{array}{ccccc}
\boxed{X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(k)} & \Longrightarrow & \boxed{X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \overline{X(k)}} & \Longrightarrow & \boxed{X(k) = \emptyset \iff X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \emptyset} \\
\Uparrow & & \Uparrow & & \Uparrow \\
\boxed{X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = X(k)} & \Longrightarrow & \boxed{X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = \overline{X(k)}} & \Longrightarrow & \boxed{X(k) = \emptyset \iff X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = \emptyset} \\
\Uparrow & & \Uparrow & & \Uparrow \\
\boxed{X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(k)} & \Longrightarrow & \boxed{X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \overline{X(k)}} & \Longrightarrow & \boxed{X(k) = \emptyset \iff X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \emptyset}
\end{array}$$

We have the indicated implications. If $X(k)$ is finite, then we obviously have $X(k) = \overline{X(k)}$, and corresponding statements in the left and middle columns are equivalent. In particular, this is the case when X is a curve of genus at least 2.

Let us discuss these statements. The ones in the middle column are perhaps the most natural ones, whereas the ones in the left column are better suited for proofs (as we will see below). The statements in the right column can be considered as variants of the Hasse Principle; in some sense they state that the Hasse Principle will eventually hold if one allows oneself to replace X by finite étale coverings. Note that the weakest of the nine statements (the one in the upper right corner), if valid for a class of varieties, would imply that there is an effective procedure to decide whether there are k-rational points on a variety X within that class or not: at least in principle, we can list all the X-torsors and for each torsor compute the finite set of twists with points everywhere locally. If this set is empty, we know that $X(k) = \emptyset$. On the other hand, we can search for k-rational points on X at the same time, and as soon as we find one such point, we know that $X(k) \neq \emptyset$. The statement "$X(k) = \emptyset \iff X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \emptyset$" guarantees that one of the two events must occur. (Note that $X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ can be written as a filtered intersection of compact subsets of $X(\mathbb{A}_k)_\bullet$, each coming from one specific torsor, so if $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \emptyset$, then already one of these conditions will provide an obstruction.)

For X of dimension at least two, none of these statements can be expected to hold in general. For example, a rational surface X has trivial geometric fundamental group, and so $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet$. On the other hand, there are examples known of such surfaces that violate the Hasse principle, so that we have $\emptyset = X(k) \subsetneq X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet$. The first example (a smooth cubic surface) was given by Swinnerton-Dyer [Sw]. There are also examples among smooth diagonal cubic surfaces, see [CG], and in [CCS], an infinite family of rational surfaces violating the Hasse principle is given.

Let us give names to the properties in the left two columns in the diagram 6.1 above.

Definition 6.1. Let X be a smooth projective k-variety. We call X

(1) *good with respect to all coverings* or simply *good* if $\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$,
(2) *good with respect to solvable coverings* if $\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$,
(3) *good with respect to abelian coverings* or *very good* if $\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$,
(4) *excellent with respect to all coverings* if $X(k) = X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$,
(5) *excellent with respect to solvable coverings* if $X(k) = X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$,
(6) *excellent with respect to abelian coverings* if $X(k) = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$.

Now let us look at curves in more detail. When C is a curve of genus 0, then it satisfies the Hasse Principle, so

$$C(\mathbb{A}_k)_\bullet = \emptyset \iff C(k) = \emptyset\,,$$

and then all the intermediate sets are equal and empty. On the other hand, when $C(k) \neq \emptyset$, then $C \cong \mathbb{P}^1$, and $C(k)$ is dense in $C(\mathbb{A}_k)_\bullet$, so

$$\overline{C(k)} = C(\mathbb{A}_k)_\bullet^{\text{f-cov}} = C(\mathbb{A}_k)_\bullet^{\text{f-sol}} = C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(\mathbb{A}_k)_\bullet\,.$$

So curves of genus 0 are always very good.

Now consider the case of a genus 1 curve. If A is an elliptic curve, or more generally, an abelian variety, then $\pi_1(\bar{A})$ is abelian and $A(k) \neq \emptyset$, so by Lemma 5.6 we have

$$A(\mathbb{A}_k)_\bullet^{\text{f-cov}} = A(\mathbb{A}_k)_\bullet^{\text{f-sol}} = A(\mathbb{A}_k)_\bullet^{\text{f-ab}}\,.$$

Furthermore, among the abelian coverings, we can restrict to the multiplication-by-n maps $A \xrightarrow{n} A$. This shows that

$$A(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \widehat{\text{Sel}}(k, A)\,.$$

Since the cokernel of the canonical map

$$\overline{A(k)} \cong \widehat{A(k)} \longrightarrow \widehat{\text{Sel}}(k, A)$$

is the Tate module of $Ш(k, A)$, we get the following.

Corollary 6.2.

$$A \textit{ is very good} \iff Ш(k, A)_{\text{div}} = 0$$
$$A \textit{ is excellent w.r.t. abelian coverings} \iff A(k) \textit{ is finite and } Ш(k, A)_{\text{div}} = 0$$

See Wang's paper [Wa] for a discussion of the situation when one works with $A(\mathbb{A}_k)$ instead of $A(\mathbb{A}_k)_\bullet$. Note that Wang's discussion is in the context of the Brauer-Manin obstruction, which is closely related to the "finite abelian" obstruction considered here, as discussed in Section 7 below.

Corollary 6.3. *If $A/\mathbb{Q}$ is a modular abelian variety of analytic rank zero, then A is excellent w.r.t. abelian coverings. In particular, if $E/\mathbb{Q}$ is an elliptic curve of analytic rank zero, then E is excellent w.r.t. abelian coverings.*

PROOF: By work of Kolyvagin [Kol] and Kolyvagin-Logatchev [KL], we know that $A(\mathbb{Q})$ and $\text{Ш}(\mathbb{Q}, A)$ are both finite. By the above, it then follows that $A(\mathbb{A}_\mathbb{Q})_\bullet^{\text{f-ab}} = A(\mathbb{Q})$.

If $E/\mathbb{Q}$ is an elliptic curve, then by work of Wiles [Wi], Taylor-Wiles [TW] and Breuil, Conrad, Diamond and Taylor [BCDT], we know that E is modular and so the first assertion applies. □

Now let X be a principal homogeneous space for the abelian variety A. If $X(\mathbb{A}_k)_\bullet = \emptyset$, then all statements in (6.1) are trivially true. So assume $X(\mathbb{A}_k)_\bullet \neq \emptyset$, and let $\xi \in \text{Ш}(k, A)$ denote the element corresponding to X. If $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \neq \emptyset$ or $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \emptyset$, we have

$$X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}},$$

and $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \emptyset$ if and only if $\xi \notin \text{Ш}(k, A)_{\text{div}}$. So for $\xi \neq 0$, X is very good if and only if $\xi \notin \text{Ш}(k, A)_{\text{div}}$ (since $X(k) = \emptyset$ in this case).

For curves C of genus 2 or higher, we always have that $C(k)$ is finite, and so the statements in the left and middle columns in 6.1 are equivalent. In this case, we can characterize the set $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ in a different way.

Theorem 6.4. *Let C be a smooth projective geometrically connected curve over k. Let $A = \text{Alb}_C^0$ be its Albanese variety, and let $V = \text{Alb}_C^1$ be the torsor under A that parametrizes classes of zero-cycles of degree 1 on C. Then there is a canonical map $\phi : C \to V$, and we have*

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \phi^{-1}(V(\mathbb{A}_k)_\bullet^{\text{f-ab}}).$$

Of course, since C is a curve, A is the same as the Jacobian variety $\text{Jac}_C = \text{Pic}_C^0$, and V is its torsor Pic_C^1, parametrizing divisor classes of degree 1 on C.

PROOF: We know by Prop. 5.8 that $\phi(C(\mathbb{A}_k)_\bullet^{\text{f-ab}}) \subset V(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. It therefore suffices to prove that $\phi^{-1}(V(\mathbb{A}_k)_\bullet^{\text{f-ab}}) \subset C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$.

By [Se3, § VI.2], all (connected) finite abelian unramified coverings of $\bar{C} = C \times_k \bar{k}$ are obtained through pull-back from isogenies into $\bar{V} \cong \bar{A}$. From this, we can deduce that the induced homomorphism $\phi^* : H^1_{\text{ét}}(\bar{V}, \bar{G}) \to H^1_{\text{ét}}(\bar{C}, \bar{G})$ is an isomorphism for all finite abelian k-group schemes G. Since the map ϕ is defined over k, we obtain an isomorphism as k-Galois modules. The spectral sequence associated to the composition of functors $H^0(k, H^0_{\text{ét}}(\bar{V}, -)) = H^0_{\text{ét}}(V, -)$ (and similarly for C) gives a diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(k,G) & \longrightarrow & H^1_{\text{ét}}(V,G) & \longrightarrow & H^0(k, H^1_{\text{ét}}(\bar{V},\bar{G})) & \longrightarrow & H^2(k,G) \\ & & \| & & \downarrow \phi^* & & \cong \downarrow \phi^* & & \| \\ 0 & \longrightarrow & H^1(k,G) & \longrightarrow & H^1_{\text{ét}}(C,G) & \longrightarrow & H^0(k, H^1_{\text{ét}}(\bar{C},\bar{G})) & \longrightarrow & H^2(k,G) \end{array}$$

By the 5-lemma, $\phi^* : H^1_{\text{ét}}(V,G) \to H^1_{\text{ét}}(C,G)$ is an isomorphism.

Let $P \in C(\mathbb{A}_k)_\bullet$ such that $\phi(P) \in V(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, and let $(Y,G) \in \mathcal{A}b(C)$. Then by the above, there is $(W,G) \in \mathcal{A}b(V)$ such that Y is the pull-back of W. By assumption, $\phi(P)$ survives (W,G); without loss of generality, (W,G) already lifts $\phi(P)$. (G is abelian, hence equal to all its inner forms.) Then (Y,G) lifts P, so P survives (Y,G). Since (Y,G) was arbitrary, $P \in C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. □

Remark 6.5. The result in the preceding theorem will hold more generally for smooth projective geometrically connected varieties X instead of curves C, provided all finite étale abelian coverings of $\bar{X}$ can be obtained as pullbacks of isogenies into the Albanese variety of X. For this, it is necessary and sufficient that the (geometric) Néron-Severi group of X is torsion-free, see [Se3, VI.20].

For arbitrary varieties X, we can define a set $X(\mathbb{A}_k)_\bullet^{\text{Alb}}$ consisting of the adelic points on X surviving all torsors that are pull-backs of V-torsors (where V is the k-torsor under A that receives a canonical map ϕ from X), and then the result above will hold in the form

$$X(\mathbb{A}_k)_\bullet^{\text{Alb}} = \phi^{-1}(V(\mathbb{A}_k)_\bullet^{\text{f-ab}}) .$$

We trivially have $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\text{Alb}}$.

In particular, we get that $X(\mathbb{A}_k)_\bullet^{\text{Alb}} = X(\mathbb{A}_k)_\bullet$ if X has trivial Albanese variety. For example, this is the case for all complete intersections of dimension at least 2 in some projective space. (By Exercise III.5.5 in [Hs], $H^1(X, \mathcal{O}) = 0$ in this case (over $\overline{k}$, say), so the Picard variety and therefore also its dual $\text{Alb}^0(X)$ are trivial.) If in addition NS_X is torsion-free, then $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet$ as well.

Corollary 6.6. *Let C be a smooth projective geometrically connected curve over k. Let A be its Albanese (or Jacobian) variety, and let $V = \text{Alb}^1_C = \text{Pic}^1_C$ as above.*

(1) *If $C(\mathbb{A}_k)_\bullet = \emptyset$, then $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(k) = \emptyset$.*

(2) *If $C(\mathbb{A}_k)_\bullet \neq \emptyset$ and $V(k) \neq \emptyset$ (i.e., C has a k-rational divisor class of degree 1), then there is a k-defined embedding $\phi : C \hookrightarrow A$, and we have*

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \phi^{-1}(\widehat{\text{Sel}}(k, A)) .$$

If $Ш(k, A)_{\text{div}} = 0$, we have

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \phi^{-1}(\overline{A(k)}) .$$

(3) *If* $C(\mathbb{A}_k)_\bullet \neq \emptyset$ *and* $V(k) = \emptyset$*, then, using the canonical map* $\phi : C \to V$*, we have*

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \phi^{-1}(V(\mathbb{A}_k)_\bullet^{\text{f-ab}}) .$$

Let $\xi \in Ш(k, A)$ *be the element corresponding to* V*. By assumption,* $\xi \neq 0$*. Then if* $\xi \notin Ш(k, A)_{\text{div}}$ *(and so in particular when* $Ш(k, A)_{\text{div}} = 0$*), we have* $C(k) = C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \emptyset$*.*

Similar statements are true for more general X *in place of* C*, with* $X(\mathbb{A}_k)_\bullet^{\text{Alb}}$ *in place of* $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$*.*

Proof: This follows immediately from Thm. 6.4, taking into account the descriptions of $A(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ and $V(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ in Cor. 6.2 and the text following it. □

Let X be a smooth projective geometrically connected k-variety, let A be its Albanese variety, and denote by V the k-torsor under A such that there is a canonical map $\phi : X \to V$. (V corresponds to the cocycle class of $\sigma \mapsto [P^\sigma - P] \in A(\bar{k})$ for any point $P \in X(\bar{k})$.) If $V(k) \neq \emptyset$, then V is the trivial torsor, and there is an n-covering of V, i.e., a V-torsor under $A[n]$. So the non-existence of an n-covering of V is an obstruction against rational points on V and therefore on X.

If an n-covering of V exists, we can pull it back to a torsor $(Y, A[n]) \in \mathcal{A}b(X)$, and we will say that a point $P \in X(\mathbb{A}_k)_\bullet$ *survives the* n*-covering of* X if it survives $(Y, A[n])$. If there is no n-covering, then by definition no point in $X(\mathbb{A}_k)_\bullet$ survives the n-covering of X. If we denote the set of adelic points surviving the n-covering of X by $X(\mathbb{A}_k)_\bullet^{n\text{-ab}}$, then we have

$$X(\mathbb{A}_k)_\bullet^{\text{Alb}} = \bigcap_{n \geq 1} X(\mathbb{A}_k)_\bullet^{n\text{-ab}} .$$

In particular, for a curve C, we get

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \bigcap_{n \geq 1} C(\mathbb{A}_k)_\bullet^{n\text{-ab}} .$$

7. Relation with the Brauer-Manin obstruction

In this section, we study the relationship between the finite covering obstructions introduced in Section 5 and the Brauer-Manin obstruction. This latter obstruction was introduced by Manin [Mn] in 1970 in order to provide a unified framework to explain violations of the Hasse Principle.

The idea is as follows. Let X be, as usual, a smooth projective geometrically connected k-variety. We then have the (cohomological) Brauer group

$$\operatorname{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m) .$$

If K/k is any field extension and $P \in X(K)$ is a K-point of X, then the corresponding morphism $\operatorname{Spec} K \to X$ induces a homomorphism $\phi_P : \operatorname{Br}(X) \to \operatorname{Br}(K)$. If $K = k_v$ is a completion of k, then there is a canonical injective homomorphism

$$\operatorname{inv}_v : \operatorname{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

(which is an isomorphism when v is a finite place). In this way, we can set up a pairing

$$X(\mathbb{A}_k)_\bullet \times \operatorname{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}, \quad ((P_v), b) \longmapsto \langle (P_v), b\rangle_{Br} = \sum_v \operatorname{inv}_v\bigl(\phi_{P_v}(b)\bigr).$$

By a fundamental result of Class Field Theory, k-rational points on X pair trivially with all elements of $\operatorname{Br}(X)$. This implies that

$$\overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \{P \in X(\mathbb{A}_k)_\bullet : \langle P, b\rangle_{\mathrm{Br}} = 0 \text{ for all } b \in \operatorname{Br}(X)\}.$$

The set $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$ is called the *Brauer set* of X. If it is empty, one says that there is a *Brauer-Manin obstruction* against rational points on X. More generally, if $B \subset \operatorname{Br}(X)$ is a subgroup (or subset), we can define $X(\mathbb{A}_k)_\bullet^B$ in a similar way as the subset of points in $X(\mathbb{A}_k)_\bullet$ that pair trivially with all $b \in B$.

The main result of this section is that for a curve C, we have

$$C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = C(\mathbb{A}_k)_\bullet^{\text{f-ab}},$$

see Cor. 7.3 below. This implies that all the results we have deduced or will deduce about finite abelian descent obstructions on curves also apply to the Brauer-Manin obstruction.

We first recall that the (algebraic) Brauer-Manin obstruction is at least as strong as the obstruction coming from finite abelian descent. For a more precise statement, see [HS1, Thm. 4.9]. We define

$$\operatorname{Br}_1(X) = \ker\bigl(\operatorname{Br}(X) \longrightarrow \operatorname{Br}(X \times_k \bar{k})\bigr) \subset \operatorname{Br}(X)$$

and set $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1(X)}$.

Theorem 7.1. *For any smooth projective geometrically connected variety X, we have*

$$X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}}.$$

PROOF: The main theorem of descent theory of Colliot-Thélène and Sansuc [CS], as extended by Skorobogatov (see [Sk1] and [Sk2, Thm. 6.1.1]), states that $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1}$ is equal to the set obtained from descent obstructions with respect to torsors under k-groups G of multiplicative type, which includes all finite abelian k-groups. This proves the second inclusion. The first one follows from the definitions. □

It is known that (see [Sk2, Cor. 2.3.9]; use that $H^3(k, \bar{k}^\times) = 0$)

$$\frac{\operatorname{Br}_1(X)}{\operatorname{Br}_0(X)} \cong H^1(k, \operatorname{Pic}_X)\,,$$

where $\operatorname{Br}_0(X)$ denotes the image of $\operatorname{Br}(k)$ in $\operatorname{Br}(X)$. We also have the canonical map $H^1(k, \operatorname{Pic}^0_X) \to H^1(k, \operatorname{Pic}_X)$. Define $\operatorname{Br}_{1/2}(X)$ to be the subgroup of $\operatorname{Br}_1(X)$ that maps into the image of $H^1(k, \operatorname{Pic}^0_X)$ in $H^1(k, \operatorname{Pic}_X)$. (Manin [Mn] calls it $\operatorname{Br}'_1(X)$.) In addition, for $n \ge 1$, let $\operatorname{Br}_{1/2,n}(X)$ be the subgroup of $\operatorname{Br}_1(X)$ that maps into the image of $H^1(k, \operatorname{Pic}^0_X)[n]$. Then

$$\operatorname{Br}_{1/2}(X) = \bigcup_{n \ge 1} \operatorname{Br}_{1/2,n}(X)$$

and

$$X(\mathbb{A}_k)_\bullet^{\operatorname{Br}_{1/2}} = \bigcap_{n \ge 1} X(\mathbb{A}_k)_\bullet^{\operatorname{Br}_{1/2,n}}\,.$$

Recall the definition of $X(\mathbb{A}_k)_\bullet^{\text{Alb}}$ from Remark 6.5 and the fact that

$$X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\text{Alb}} = \bigcap_{n \ge 1} X(\mathbb{A}_k)_\bullet^{n\text{-ab}}\,.$$

Theorem 7.2. *Let X be a smooth projective geometrically connected variety, and let $n \ge 1$. Then*

$$X(\mathbb{A}_k)_\bullet^{n\text{-ab}} \subset X(\mathbb{A}_k)_\bullet^{\operatorname{Br}_{1/2,n}}\,.$$

In particular,

$$X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\text{Alb}} \subset X(\mathbb{A}_k)_\bullet^{\operatorname{Br}_{1/2}}\,.$$

Proof: Given the first statement, the second statement is clear.

So consider $P \in X(\mathbb{A}_k)_\bullet^{n\text{-ab}}$ and $b \in \operatorname{Br}_{1/2,n}(X)$. We have to show that $\langle b, P\rangle_{\operatorname{Br}} = 0$, where $\langle \cdot, \cdot \rangle_{\operatorname{Br}}$ is the Brauer pairing between $X(\mathbb{A}_k)_\bullet$ and $\operatorname{Br}(X)$.

Let b' be the image of b in $\operatorname{Br}_1(X)/\operatorname{Br}_0(X) \cong H^1(k, \operatorname{Pic}_X)$, and let $b'' \in H^1(k, \operatorname{Pic}^0_X)[n]$ be an element mapping to b' (which exists since $b \in \operatorname{Br}_{1/2,n}(X)$).

Let A be the Albanese variety of X, and let V be the k-torsor under A that has a canonical map $\phi : X \to V$. Then we have $\operatorname{Pic}^0_X \cong \operatorname{Pic}^0_A \cong \operatorname{Pic}^0_V$. Since $P \in X(\mathbb{A}_k)_\bullet^{n\text{-ab}} \xrightarrow{\phi} V(\mathbb{A}_k)_\bullet^{n\text{-ab}}$, the latter is nonempty, hence V admits a torsor of the form $(W, A[n])$.

Since P maps into $V(\mathbb{A}_k)_\bullet^{n\text{-ab}}$, there is some twist of $(W, A[n])$ such that $\phi(P)$ lifts to it. Without loss of generality, $(W, A[n])$ is already this twist, so there is $Q' \in W(\mathbb{A}_k)_\bullet$ such that $\pi'(Q') = \phi(P)$, where $\pi' : W \to V$ is the covering map associated to $(W, A[n])$.

Let $(Y, A[n]) \in \mathcal{A}b(X)$ be the pull-back of $(W, A[n])$ to X. Then there is some $Q \in Y(\mathbb{A}_k)_\bullet$ such that $\pi(Q) = P$. Now the left hand diagram below induces the one on the right, where the rightmost vertical map is multiplication by n:

$$\begin{array}{ccc} Y & \longrightarrow & W \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{\ \phi\ } & V \end{array} \qquad\qquad \begin{array}{ccccccc} \mathrm{Pic}_Y & \longleftarrow & \mathrm{Pic}^0_Y & \longleftarrow & \mathrm{Pic}^0_W & = & \mathrm{Pic}^0_A \\ {\scriptstyle \pi^*}\uparrow & & {\scriptstyle \pi^*}\uparrow & & {\scriptstyle \pi'^*}\uparrow & & \uparrow{\scriptstyle \cdot n} \\ \mathrm{Pic}_X & \longleftarrow & \mathrm{Pic}^0_X & \xleftarrow{\ \cong\ } & \mathrm{Pic}^0_V & = & \mathrm{Pic}^0_A \end{array}$$

Chasing b'' around the diagram on the right, after applying $H^1(k, -)$ to it, we see that $\pi^*(b') = 0$ in $\mathrm{Br}(Y)/\mathrm{Br}_0(Y)$. Finally, we have

$$\langle b, P\rangle_{\mathrm{Br}} = \langle b', \pi(Q)\rangle_{\mathrm{Br}} = \langle \pi^*(b'), Q\rangle_{\mathrm{Br}} = 0\,.$$

□

So we have the chain of inclusions

$$X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}\,.$$

It is then natural to ask to what extent one might have equality in this chain of inclusions. We certainly get something when $\mathrm{Br}_{1/2}(X)$ already equals $\mathrm{Br}_1(X)$ or even $\mathrm{Br}(X)$.

Corollary 7.3. *If X is a smooth projective geometrically connected variety such that the canonical map $H^1(k, \mathrm{Pic}^0_X) \to H^1(k, \mathrm{Pic}_X)$ is surjective, then*

$$X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}}\,.$$

In particular, if C is a curve, then $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$.

Proof: In this case, $\mathrm{Br}_{1/2}(X) = \mathrm{Br}_1(X)$, and so the result follows from the two preceding theorems.

When $X = C$ is a curve, then we know that $\mathrm{Br}(C \times_k \bar{k})$ is trivial (Tsen's Theorem); also $H^1(k, \mathrm{Pic}^0_C)$ surjects onto $H^1(k, \mathrm{Pic}_C)$, since the Néron-Severi group of C is $\mathbb{Z}$ with trivial Galois action, and $H^1(k, \mathbb{Z}) = 0$. Hence $\mathrm{Br}(C) = \mathrm{Br}_{1/2}(C)$, and the assertion follows. □

The result of Cor. 7.3 means that we can replace $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ by $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$ everywhere. For example, from Cor. 6.6, we obtain the following.

Corollary 7.4. *Let C be a smooth projective geometrically connected curve over k, and Let A be its Albanese (or Jacobian) variety. Assume that $Ш(k, A)_{\mathrm{div}} = 0$.*

(1) *If C has a k-rational divisor class of degree 1 inducing a k-defined embedding $C \hookrightarrow A$, then*

$$C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \phi^{-1}(\overline{A(k)})\,,$$

where ϕ denotes the induced map $C(\mathbb{A}_k)_\bullet \to A(\mathbb{A}_k)_\bullet$.

(2) *If C has no k-rational divisor class of degree 1, then $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \emptyset$.*

These results can be found in Scharaschkin's thesis [Sc]. Our approach provides an alternative proof, and the more precise version in Cor. 6.6 shows how to extend the result to the case when the Shafarevich-Tate group of the Jacobian is not necessarily assumed to have trivial divisible subgroup.

I claim that we actually have equality in Thm. 7.2. Let us first prove the reverse inclusion in the case that X is a k-torsor under an abelian variety A. Let $A^\vee = \mathrm{Pic}^0_X$ be the dual abelian variety. We assume that $X(\mathbb{A}_k)_\bullet \neq \emptyset$; otherwise the claim is trivially true. We then have the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & H^2(k, \bar{k}(X)^\times) & \longrightarrow & H^2(k, \bar{k}(X)^\times/\bar{k}^\times) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & \mathrm{Br}_1(X) & \longrightarrow & H^1(k, \mathrm{Pic}_X) & \longrightarrow & 0 \end{array}$$

and the corresponding local versions. To obtain the Brauer pairing $\langle P, b\rangle_{\mathrm{Br}}$ for $P \in X(\mathbb{A}_k)_\bullet$ and $b \in \mathrm{Br}_{1/2}(X)$, we map b_v to $f_v \in H^2(k_v, \bar{k}_v(X)^\times)$ and evaluate at P_v (the fact that f_v is in the image of $\mathrm{Br}(X_v)$ guarantees that we can do that, compare [Li]) to get $f_v(P_v) \in H^2(k_v, \bar{k}_v^\times) = \mathrm{Br}(k_v)$. The pairing is then

$$\langle P, b\rangle_{\mathrm{Br}} = \sum_v \mathrm{inv}_v\, f_v(P_v)\,,$$

see [Mn, p. 410]. If b maps to β in the image of $Ш(k, A^\vee)$ in $H^1(k, \mathrm{Pic}_X)$, then $\langle P, b\rangle_{\mathrm{Br}}$ equals the Cassels-Tate pairing of X (as an element of $Ш(k, A)$) and β, see Remark 6.11 in [Mi] or Manin's ICM address [Mn] or also Thm. 6.2.3 in [Sk2]. In particular, the pairing does not depend on P.

Lemma 7.5. *Suppose that X does not have an n-covering with points everywhere locally. Then $X(\mathbb{A}_k)_\bullet^{n\text{-ab}}$ and $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}}$ are both empty.*

Proof: If X does not have an n-covering with points everywhere locally, then $X(\mathbb{A}_k)_\bullet^{n\text{-ab}} = \emptyset$ by definition. Also, X is not divisible by n as an element of $Ш(k, A)$, hence there is an element $\beta \in Ш(k, A^\vee)[n]$ that pairs non-trivially with X under the Cassels-Tate pairing. Let $b \in \mathrm{Br}_{1/2,n}(X)$ be an element having the same image in $H^1(k, \mathrm{Pic}_X)$ as β. Then $\langle P, b\rangle_{\mathrm{Br}}$ does not depend on P and is nonzero, hence $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}} \subset X(\mathbb{A}_k)_\bullet^{b} = \emptyset$ as well. □

So from now on, we can assume that X does have an n-covering $\pi : Y \to X$ such that $Y(\mathbb{A}_k)_\bullet \neq \emptyset$.

Consider the pairing

$$\begin{aligned} \langle\cdot,\cdot\rangle_1 : {\prod_v}' H^1(k_v, A[n]) \times \bigoplus_v H^1(k_v, A^\vee[n]) &\longrightarrow \mathbb{Q}/\mathbb{Z}\,, \\ ((\alpha_v)_v, (\beta_v)_v) &\longmapsto \sum_v \operatorname{inv}_v(\alpha_v \cup_e \beta_v)\,, \end{aligned}$$

where $\cup_e$ is the cup product pairing taken with respect to the Weil pairing $e_n : A[n] \times A^\vee[n] \to \mu_n$. Here $\prod'$ is a restricted product, whose elements have their v-component in the unramified cohomology for all but finitely many v. By a simple special case of Lemma 6.15 in [Mi] (which is a consequence of Thm. 4.10 in the same reference), the annihilator under this pairing of the image of $H^1(k, A^\vee[n])$ in $\bigoplus_v H^1(k_v, A^\vee[n])$ is the image of $H^1(k, A[n])$ in $\prod'_v H^1(k_v, A[n])$.

Using the n-covering $Y \to X$, we can define another pairing

$$\langle\cdot,\cdot\rangle_2 : X(\mathbb{A}_k)_\bullet \times H^1(k, A^\vee[n]) \longrightarrow \mathbb{Q}/\mathbb{Z}\,, \qquad ((P_v)_v, \xi) \longmapsto \langle (d_Y P_v)_v, (\xi_v)_v\rangle_1\,,$$

where $d_Y P_v$ is the class of the cocycle $\sigma \mapsto Q_v^\sigma - Q_v$, with Q_v any preimage of P_v in $Y(\bar{k}_v)$. Note that this pairing is independent of the choice of Y: any other choice Y' will have $d_{Y'} P_v = d_Y P_v + \alpha_v$, where $\alpha \in H^1(k, A[n])$ is the class which one twist Y by to get Y'. By the above, α annihilates ξ, so the value of the pairing is the same. If $Q = (Q_v)_v \in Y(\mathbb{A}_k)_\bullet$, then $d_Y \pi(Q_v) = 0$, and therefore

$$\langle \pi(Q), \xi\rangle_2 = 0 \quad \text{for all } \xi \in H^1(k, A^\vee[n]).$$

Now let $\beta \in H^1(k, A^\vee)[n]$ and $P \in X(\mathbb{A}_k)_\bullet$. We define

$$\langle P, \beta\rangle_3 = \langle P, \xi\rangle_2\,,$$

where $\xi \in H^1(k, A^\vee[n])$ is any lift of β. Since the difference of any two possible lifts is in the image of $A^\vee(k)$ and images of points pair trivially under $\cup_e$, this is again well-defined. We then have

$$\langle P, \beta\rangle_3 = 0 \text{ for all } \beta \in H^1(k, A^\vee[n]) \iff P \in X(\mathbb{A}_k)_\bullet^{n\text{-ab}} \tag{7.1}$$

("$\Longleftarrow$": This is clear from the above when Y lifts P. Otherwise replace Y by the twist of Y lifting P and remember that the pairing does not depend on the choice of Y.

"$\Longrightarrow$": By the remark above, $d_Y P$ is in the image of $H^1(k, A[n])$. Replacing Y by the corresponding twist, we can assume that $d_Y P = 0$. But this means that Y lifts P.)

Our goal now is to show that

$$\langle P, \beta\rangle_3 = \langle P, b\rangle_{\mathrm{Br}}$$

for $P \in X(\mathbb{A}_k)_\bullet$, $b \in \mathrm{Br}_{1/2,n}(X)$, $\beta \in H^1(k, A^\vee)[n]$ such that b and β have the same image in $H^1(k, \mathrm{Pic}_X)$. Recall that

$$\langle P, b\rangle_{\mathrm{Br}} = \sum_v \mathrm{inv}_v\, f_v(P_v)\,,$$

where f is the image of b in $H^2(k, \bar{k}(X)^\times)$.

Now, if $P_v \in A(k_v)$ and $\beta_v \in H^1(k_v, A^\vee)[n]$, then the local Tate duality pairing $\langle P_v, \beta_v\rangle_{\mathrm{Tate},v}$ can be computed on the one hand as $dP_v \cup_e \xi_v$, where dP_v is the image of P_v under the coboundary map $A(k_v) \to H^1(k_v, A[n])$ and ξ_v is a lift of β_v to $H^1(k_v, A^\vee[n])$. On the other hand (at least up to a sign, which need not concern us), it can be obtained by evaluating the image of β_v in $H^2(k_v, \bar{k}_v(X)^\times/\bar{k}_v^\times)$ on a zero-cycle of degree zero on X_v summing to P_v — note that over k_v, X and A are isomorphic. (Compare Remark 3.5 in [Mi].)

Now fix $Q = (Q_v)_v \in Y(\mathbb{A}_k)_\bullet$, and let $\pi : Y \to X$ be the covering map. Over k_v, we have isomorphisms $\phi : Y \to A$, $R_v \mapsto R_v - Q_v$ and $\psi : X \to A$, $P_v \mapsto P_v - \pi(Q_v)$, such that the following diagrams commute.

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} \\ A & \xrightarrow{n} & A \end{array} \qquad\qquad \begin{array}{ccc} X(k_v) & \xrightarrow{d_Y} & H^1(k_v, A[n]) \\ {\scriptstyle\psi}\downarrow & & \| \\ A(k_v) & \xrightarrow{d} & H^1(k_v, A[n]) \end{array}$$

This implies that

$$\langle (P_v)_v, \beta\rangle_3 = \langle (P_v - \pi(Q_v))_v, \beta\rangle_{\mathrm{Tate}}\,,$$

where on the right hand side, we have the global Tate duality pairing. Let now $f \in H^2(k, \bar{k}(X)^\times)$ be the image of b, and let $f' \in H^2(k, \bar{k}(X)^\times/\bar{k}^\times)$ be the common image of b and β. Then, with $[P_v] - [\pi(Q_v)]$ denoting the obvious zero-cycle of degree zero on X_v,

$$\begin{aligned}
\langle P, \beta\rangle_3 &= \langle (P_v - \pi(Q_v))_v, \beta\rangle_{\mathrm{Tate}} \\
&= \sum_v \mathrm{inv}_v\, f'_v([P_v] - [\pi(Q_v)]) \\
&= \sum_v \mathrm{inv}_v\, f_v([P_v] - [\pi(Q_v)]) \\
&= \sum_v \mathrm{inv}_v\, f_v(P_v) - \sum_v \mathrm{inv}_v\, f_v(\pi(Q_v)) \\
&= \langle P, b\rangle_{\mathrm{Br}} - \langle \pi(Q), b\rangle_{\mathrm{Br}} \\
&= \langle P, b\rangle_{\mathrm{Br}}\,,
\end{aligned}$$

since $\langle \pi(Q), b\rangle_{\mathrm{Br}}$ vanishes by Thm. 7.2. See also [Mn, Prop. 8] for the connection between the Brauer and Tate pairings.

We now have almost proved the following lemma.

Lemma 7.6. *Assume that X has an n-covering Y such that $Y(\mathbb{A}_k)_\bullet \neq \emptyset$. Then*

$$X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}} \subset X(\mathbb{A}_k)_\bullet^{n\text{-ab}}.$$

PROOF: Let $P \in X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}}$. By the above, we then have $\langle P, \beta\rangle_3 = 0$ for all $\beta \in H^1(k, A^\vee)[n]$. The equivalence in (7.1) then tells us that $P \in X(\mathbb{A}_k)_\bullet^{n\text{-ab}}$. □

Now we let X be an arbitrary smooth projective geometrically connected variety again.

Theorem 7.7. *Let X be a smooth projective geometrically connected variety. Then*

$$X(\mathbb{A}_k)_\bullet^{n\text{-ab}} = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}}$$

for all $n \geq 1$. In particular,

$$X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}} = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}.$$

PROOF: Let A be the Albanese variety of X and V the k-torsor under A that receives a canonical map $\phi : X \to V$. By Lemmas 7.5 and 7.6 above, we have

$$V(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}} \subset V(\mathbb{A}_k)_\bullet^{n\text{-ab}}.$$

Since $X(\mathbb{A}_k)_\bullet^{n\text{-ab}} = \phi^{-1}(V(\mathbb{A}_k)_\bullet^{n\text{-ab}})$, we find

$$X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}} \subset \phi^{-1}(V(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}}) \subset \phi^{-1}(V(\mathbb{A}_k)_\bullet^{n\text{-ab}}) = X(\mathbb{A}_k)_\bullet^{n\text{-ab}}.$$

The reverse inclusion is provided by Thm. 7.2. □

Remark 7.8. Since $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}$, it is natural to ask whether there might be a subgroup $B \subset \mathrm{Br}_1(X)$ such that $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet^{B}$. As Joost van Hamel pointed out to me, a natural candidate for B is the subgroup mapping to the image of $H^1(k, \mathrm{Pic}_X^\tau)$ in $H^1(k, \mathrm{Pic}_X)$, where Pic_X^τ is the saturation of Pic_X^0 in Pic_X, i.e., the subgroup of elements mapping into the torsion subgroup of the Néron-Severi group NS_X. It is tempting to denote this B by $\mathrm{Br}_{2/3}$, but perhaps Br_τ is the better choice. Note that $\mathrm{Br}_\tau = \mathrm{Br}_{1/2}$ when NS_X is torsion free, in which case we have $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}} = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}$. Also, the torsion subgroup of NS_X is associated to an abelian covering of X, which makes it likely that information coming from this covering is related to obstructions coming from Br_τ.

Corollary 7.9. *If C/k is a curve that has a rational divisor class of degree 1, then*

$$C(\mathbb{A}_k)_\bullet^{n\text{-ab}} = C(\mathbb{A}_k)_\bullet^{\mathrm{Br}[n]}.$$

In words, the information coming from n-torsion in the Brauer group is exactly the information obtained by an n-descent on C.

PROOF: Under the given assumptions, $H^1(k, \operatorname{Pic}^0_C) = H^1(k, \operatorname{Pic}_C) = \operatorname{Br}(C)/\operatorname{Br}(k)$, and $\operatorname{Br}(k)$ is a direct summand of $\operatorname{Br}(C)$. Therefore, the images of $\operatorname{Br}_{1/2,n}(C)$ and of $\operatorname{Br}(C)[n]$ in $\operatorname{Br}(C)/\operatorname{Br}_0(C)$ agree, and the claim follows. ◻

Corollary 7.10. *If X is a smooth projective geometrically connected variety such that the Néron-Severi group of X (over $\bar{k}$) is torsion-free, then there is a finite field extension K/k such that*

$$X(\mathbb{A}_K)^{\mathrm{Br}_1}_\bullet = X(\mathbb{A}_K)^{\text{f-ab}}_\bullet .$$

PROOF: We have an exact sequence

$$H^1(k, \operatorname{Pic}^0_X) \longrightarrow H^1(k, \operatorname{Pic}_X) \longrightarrow H^1(k, \mathrm{NS}_X) .$$

Since NS_X is a finitely generated abelian group, the Galois action on it factors through a finite quotient $\operatorname{Gal}(K/k)$ of the absolute Galois group of k. Then $H^1(K, \mathrm{NS}_X) = \operatorname{Hom}(G_K, \mathbb{Z}^r) = 0$, and the claim follows from Thm. 7.2. ◻

Note that it is not true in general that $X(\mathbb{A}_k)^{\mathrm{Br}_1}_\bullet = X(\mathbb{A}_k)^{\text{f-ab}}_\bullet$ (even when the Néron-Severi group of X over $\bar{k}$ is torsion-free). For example, a smooth cubic surface X in $\mathbb{P}^3$ has $X(\mathbb{A}_k)^{\text{f-cov}}_\bullet = X(\mathbb{A}_k)_\bullet$ (since it has trivial geometric fundamental group), but may well have $X(\mathbb{A}_k)^{\mathrm{Br}_1}_\bullet = \emptyset$, even though there are points everywhere locally. See [CKS], where the algebraic Brauer-Manin obstruction is computed for all smooth diagonal cubic surfaces

$$X : a_1\, x_1^3 + a_2\, x_2^3 + a_3\, x_3^3 + a_4\, x_4^3 = 0$$

with integral coefficients $0 < a_i < 100$, thereby verifying that it is the only obstruction against rational points on X (and thus providing convincing experimental evidence that this may be true for smooth cubic surfaces in general). This computation produces a list of 245 such surfaces with points everywhere locally, but no rational points, since $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}_1}_\bullet = \emptyset$.

It is perhaps worth mentioning that our condition that $H^1(k, \operatorname{Pic}^0_X)$ surjects onto $H^1(k, \operatorname{Pic}_X)$, which leads to the identification of the "algebraic Brauer-Manin obstruction" and the "finite abelian descent obstruction", is in some sense orthogonal to the situation studied (quite successfully) in [CS, CCS, CSS], where it is assumed that Pic_X is torsion-free (and therefore Pic^0_X is trivial), and so there can only be a Brauer-Manin obstruction when our condition fails. There is then no finite abelian descent obstruction, and one has to look at torsors under tori instead.

In general, we have a diagram of inclusions:

$$X(k) \subset \overline{X(k)} \begin{array}{c} \subset\ X(\mathbb{A}_k)^{\mathrm{Br}}_\bullet \subset X(\mathbb{A}_k)^{\mathrm{Br}_1}_\bullet\ \subset \\ \subset\ X(\mathbb{A}_k)^{\text{f-cov}}_\bullet \subset X(\mathbb{A}_k)^{\text{f-sol}}_\bullet\ \subset \end{array} X(\mathbb{A}_k)^{\text{f-ab}}_\bullet \subset X(\mathbb{A}_k)^{\mathrm{Br}_{1/2}}_\bullet \subset X(\mathbb{A}_k)_\bullet$$

We expect that every inclusion can be strict. We discuss them in turn.

(1) $X = \mathbb{P}^1$ has $X(k) \subsetneq \overline{X(k)} = X(\mathbb{A}_k)_\bullet$.

(2) Skorobogatov's famous example (see [Sk1] and [HS1]) has $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} \neq \emptyset$, but $X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = \emptyset$, showing that $\overline{X(k)} \subsetneq X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$ and $X(\mathbb{A}_k)_\bullet^{\text{f-sol}} \subsetneq X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ are both possible.

(3) As mentioned above, [CKS] has examples such that $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} = \emptyset$, but $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet$. This shows that both $\overline{X(k)} \subsetneq X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ and $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} \subsetneq X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ are possible.

(4) Harari [Ha1] has examples, where there is a "transcendental", but no "algebraic" Brauer-Manin obstruction, which means that $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \emptyset$, but $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} \neq \emptyset$. Hence we can have $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} \subsetneq X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1}$.

(5) If we take a finite nonabelian simple group for $\pi_1(\bar{X})$ in Cor. 6.1 in [Ha2], then the proof of this result shows that $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subsetneq X(\mathbb{A}_k)_\bullet$. On the other hand, $X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = X(\mathbb{A}_k)_\bullet$, since there are only trivial torsors in $\mathcal{S}ol(X)$, compare Lemma 5.6.

(6) It is likely that a construction using Enriques Surfaces like that in [HS2] can produce an example such that $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}} = X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}} = X(\mathbb{A}_k)_\bullet$, since the Albanese variety is trivial, but $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subsetneq X(\mathbb{A}_k)_\bullet$, since there is a nontrivial abelian covering.

(7) Finally, in Section 8 below, we will see many examples of curves X that have $X(k) = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}} \subsetneq X(\mathbb{A}_k)_\bullet$.

A new obstruction?

For curves, we expect the interesting part of the diagram of inclusions above to collapse: $\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}$, see the discussion in Section 9 below. For higher-dimensional varieties, this is far from true, see the discussion above. So one could consider a new obstruction obtained from a combination of the Brauer-Manin and the finite descent obstructions, as follows. Define

$$X(\mathbb{A}_k)_\bullet^{\text{f-cov,Br}} = \bigcap_{(Y,G)\in\mathcal{C}ov(X)} \bigcup_{\xi\in H^1(k,G)} \pi_\xi\Big(Y_\xi(\mathbb{A}_k)_\bullet^{\mathrm{Br}}\Big)\,.$$

(This is similar in spirit to the "refinement of the Manin obstruction" introduced in [Sk1].)

It would be interesting to find out how strong this obstruction is and whether it is strictly weaker than the obstruction obtained from *all* torsors under (not necessarily finite or abelian) k-group schemes. Note that the latter is at least as strong as the Brauer-Manin obstruction by [HS1, Thm. 4.10] (see also Prop. 5.3.4 in [Sk2]), at least if one assumes that all elements of $\mathrm{Br}(X)$ are represented by Azumaya algebras over X.

8. Finite descent conditions on curves

Let us now prove some general properties of the notions, introduced in Section 6 above, of being excellent w.r.t. all, solvable, or abelian coverings in the case of curves. In the following, C, D, etc., will be (smooth projective geometrically connected) curves over k. ι will denote an embedding of C into its Jacobian (if it exists). Also, if $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \emptyset$ (and therefore $C(k) = \emptyset$, too), we say that *the absence of rational points is explained by the Brauer-Manin obstruction.* Note that by Cor. 7.3, $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, which implies that the absence of rational points is explained by the Brauer-Manin obstruction when C is excellent w.r.t. abelian coverings and $C(k) = \emptyset$. We will use this observation below without explicit mention.

Corollary 8.1. *Let C/k be a curve of genus at least 1, with Jacobian J. Assume that $Ш(k,J)_{\mathrm{div}} = 0$ and that $J(k)$ is finite. Then C is excellent w.r.t. abelian coverings. If $C(k) = \emptyset$, the absence of rational points is explained by the Brauer-Manin obstruction.*

Proof: By Cor. 6.6, under the assumption on $Ш(k,J)$, either $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \emptyset$, and there is nothing to prove, or else

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \iota^{-1}(\overline{J(k)}) = \iota^{-1}(J(k)) = C(k)\,.$$

□

The following result shows that the statement we would like to have (namely that $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(k)$) holds for finite subschemes of a curve.

Theorem 8.2. *Let C/k be a curve of genus at least 1, and let $Z \subset C$ be a finite subscheme. Then the image of $Z(\mathbb{A}_k)_\bullet$ in $C(\mathbb{A}_k)_\bullet$ meets $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ in $Z(k)$. More generally, if $P \in C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ is such that $P_v \in Z(k_v)$ for a set of places v of k of density 1, then $P \in Z(k)$.*

Proof: Let K/k be a finite extension such that C has a rational divisor class of degree 1 over K. By Cor. 6.6, we have that

$$C(\mathbb{A}_K)_\bullet^{\text{f-ab}} = \iota^{-1}(\widehat{\mathrm{Sel}}(K,J))\,,$$

where $\iota : C(\mathbb{A}_K)_\bullet \to J(\mathbb{A}_K)_\bullet$ is the map induced by an embedding $C \hookrightarrow J$ over K. Now we apply Thm. 3.11 to the image of Z in J. We find that $\iota(P) \in \widehat{\mathrm{Sel}}(K,J)$ and so $\iota(P) \in \iota(Z(K))$. Since ι is injective (even at the infinite places!), we find that the image of P in $C(\mathbb{A}_K)_\bullet$ is in (the image of) $Z(k)$. Now if $Z(k)$ is empty, this gives a contradiction and proves the claim in this case. Otherwise, $C(k) \supset Z(k)$ is non-empty, and we can take $K = k$ above, which gives the statement directly.
□

The following results show that the "excellence properties" behave nicely.

Proposition 8.3. *Let K/k be a finite extension, and let C/k be a curve of genus at least 1. If C_K is excellent w.r.t. all, solvable, or abelian coverings, then so is C.*

PROOF: By Prop. 5.12, we have

$$C(k) \subset C(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset C(\mathbb{A}_k)_\bullet \cap C(\mathbb{A}_K)_\bullet^{\text{f-cov}} = C(\mathbb{A}_k)_\bullet \cap C(K) = C(k) \, .$$

Similarly for $C(\mathbb{A}_k)_\bullet^{\text{f-sol}}$ and $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. Strictly speaking, this means that $C(k)$ and $C(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ have the same image in $C(\mathbb{A}_K)_\bullet$. Now, since $C(K)$ has to be finite in order to equal $C(\mathbb{A}_K)_\bullet^{\text{f-cov}}$, $C(k)$ is also finite, and we can apply Thm. 8.2 to $Z = C(k) \subset C$ and the set of finite places of k. □

Proposition 8.4. *Let $(D, G) \in \mathcal{C}ov(C)$ (or $\mathcal{S}ol(C)$). If all twists D_ξ of (D, G) are excellent w.r.t. all (resp., solvable) coverings, then C is excellent w.r.t. all (resp., solvable) coverings.*

PROOF: By Thm. 5.1, $C(k) = \coprod_\xi \pi_\xi(D_\xi(k))$. Now, by Prop. 5.13,

$$C(k) \subset C(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \bigcup_\xi \pi_\xi\big(D_\xi(\mathbb{A}_k)_\bullet^{\text{f-cov}}\big) = \bigcup_\xi \pi_\xi(D_\xi(k)) = C(k) \, .$$

If G is solvable, the same proof shows the statement for $C(\mathbb{A}_k)_\bullet^{\text{f-sol}}$. □

Proposition 8.5. *Let $C \xrightarrow{\phi} X$ be a non-constant morphism over k from the curve C into a variety X. If X is excellent w.r.t. all, solvable, or abelian coverings, then so is C. In particular, if $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(k)$ and $C(k) = \emptyset$, then the absence of rational points on C is explained by the Brauer-Manin obstruction.*

PROOF: First assume that C is of genus zero. Then either $C(\mathbb{A}_k)_\bullet = \emptyset$, and there is nothing to prove, or else $C(k)$ is dense in $C(\mathbb{A}_k)_\bullet$, implying that $X(k) \subsetneq \overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ and thus contradicting the assumption.

Now assume that C is of genus at least 1. Let $P \in C(\mathbb{A}_k)_\bullet^{\text{f-cov/f-sol/f-ab}}$. Then by Thm. 5.8, $\phi(P) \in X(\mathbb{A}_k)_\bullet^{\text{f-cov/f-sol/f-ab}} = X(k)$. Let $Z \subset C$ be the preimage (subscheme) of $\phi(P) \in X(k)$ in C. This is finite, since ϕ is non-constant. Then we have that P is in the image of $Z(\mathbb{A}_k)_\bullet$ in $C(\mathbb{A}_k)_\bullet$. Now apply Thm. 8.2 to conclude that

$$P \in C(\mathbb{A}_k)_\bullet^{\text{f-ab}} \cap Z(\mathbb{A}_k)_\bullet = Z(k) \subset C(k) \, .$$

□

As an application, we have the following.

Corollary 8.6. *Let $C \to A$ be a non-constant morphism over k of a curve C into an abelian variety A. Assume that $\text{Ш}(k, A)_{\text{div}} = 0$ and that $A(k)$ is finite. (For example, this is the case when $k = \mathbb{Q}$ and A is modular of analytic rank zero.) Then C is excellent w.r.t. abelian coverings. In particular, if $C(k) = \emptyset$, then the absence of rational points on C is explained by the Brauer-Manin obstruction.*

PROOF: By Cor. 6.2, we have $A(\mathbb{A}_k)_\bullet^{\text{f-ab}} = A(k)$. Now by Prop. 8.5, the claim follows. □

This generalizes a result proved by Siksek [Si] under additional assumptions on the Galois action on the fibers of ϕ above k-rational points of A, in the case that $C(k)$ is empty. A similar observation was made independently by Colliot-Thélène [CT]. Note that both previous results are in the context of the Brauer-Manin obstruction.

Examples 8.7. We can use Cor. 8.6 to produce many examples of curves C over $\mathbb{Q}$ that are excellent w.r.t. abelian coverings. Concretely, let us look at the curves $C_a : y^2 = x^6 + a$, where a is a non-zero integer. C_a maps to the two elliptic curves $E_a : y^2 = x^3 + a$ and E_{a^2} (the latter by sending (x, y) to $(a/x^2, ay/x^3)$). So whenever one of these elliptic curves has (analytic) rank zero, we know that C_a is excellent w.r.t. abelian coverings. For example, this is the case for all a such that $|a| \leq 20$, with the exception of $a = -15, -13, -11, 3, 10, 11, 15, 17$. Note that $C_a(\mathbb{Q})$ is always non-empty (there are two rational points at infinity).

We can even show a whole class of interesting curves to be excellent w.r.t. abelian coverings.

Corollary 8.8. *If $C/\mathbb{Q}$ is one of the modular curves $X_0(N)$, $X_1(N)$, $X(N)$ and such that the genus of C is positive, then C is excellent w.r.t. abelian coverings.*

PROOF: By a result of Mazur [Mz], every Jacobian $J_0(p)$ of $X_0(p)$, where $p = 11$ or $p \geq 17$ is prime, has a nontrivial factor of analytic rank zero. Also, if $M \mid N$, then there are nonconstant morphisms $X_1(N) \to X_0(N) \to X_0(M)$, hence the assertion is true for all $X_0(N)$ and $X_1(N)$ such that N is divisible by one of the primes in Mazur's result. For the other minimal N such that $X_0(N)$ (resp., $X_1(N)$) is of positive genus, William Stein's tables [Ste] prove that there is a factor of $J_0(N)$ (resp., $J_1(N)$) of analytic rank zero. So we get the result for all $X_0(N)$ and $X_1(N)$ of positive genus. Finally, $X(N)$ maps to $X_0(N^2)$, and so we obtain the result also for $X(N)$ (except in the genus zero cases $N = 1, 2, 3, 4, 5$). □

For another example, involving high-genus Shimura curves, see [Sk3].

9. Some conjectures

In the preceding section, we have seen that we can construct many examples of higher-genus curves that are excellent w.r.t. abelian coverings. This leads us to state the following conjecture.

Conjecture 9.1 (Main Conjecture)**.** *If C is a smooth projective geometrically connected curve over a number field k, then C is very good.*

By what we have seen, for curves of genus 1, this is equivalent to saying that the divisible subgroup of Ш(k, E) is trivial, for every elliptic curve E over k. For curves C of higher genus, the statement is equivalent to saying that C is excellent w.r.t. abelian coverings. We recall that our conjecture would follow in this case from the "Adelic Mordell Lang Conjecture" formulated in Question 3.12.

Remark 9.2. When k is a global function field of characteristic p, then the Main Conjecture holds when $J = \mathrm{Jac}_C$ has no constant factor and $J(k^{\mathrm{sep}})[p^\infty]$ is finite. See recent work by Poonen and Voloch [PV].

For the purposes of the following discussion, we state a variant of the conjecture.

Conjecture 9.3 (Main Conjecture, weak form)**.** *If C is a smooth projective geometrically connected curve over a number field k, then C is good.*

The difference is that for higher genus curves C, we only require that C is excellent w.r.t. all coverings.

If either version of the Main Conjecture holds for C and $C(k)$ is empty, then (as previously discussed) we can *prove* that $C(k)$ is empty by exhibiting a torsor that has no twists with points everywhere locally. The validity of one of these conjectures (even just in case $C(k)$ is empty) therefore implies that *we can algorithmically decide whether a given smooth projective geometrically connected curve over a number field k has rational points or not.*

In Section 7 above, we have shown that for a curve C, we have

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(\mathbb{A}_k)_\bullet^{\mathrm{Br}},$$

where on the right hand side, we have the *Brauer subset* of $C(\mathbb{A}_k)_\bullet$, i.e., the subset cut out by conditions coming from the Brauer group of C. One says that there is a *Brauer-Manin obstruction* against rational points on C if $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \emptyset$. A corollary of the strong form of our Main Conjecture is that the Brauer-Manin obstruction is the only obstruction against rational points on curves over number fields (which means that $C(k) = \emptyset$ implies $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \emptyset$). To our knowledge, before this work (and Poonen's heurstic, see his conjecture below, which was influenced by discussions we had at the IHP in Paris in Fall 2004) nobody gave a conjecturally positive answer to the question, first formulated on page 133 in [Sk2], whether the Brauer-Manin obstruction might be the only obstruction against rational points on curves. No likely counter-example is known, but there is an ever-growing list of examples, for which the failure of the Hasse Principle could be explained by the Brauer-Manin obstruction; see the discussion below (which does not pretend to be exhaustive) or also Skorobogatov's recent paper [Sk3] on Shimura curves.

Let v be a place of k. Under a *local condition at v* on a rational point $P \in C$, we understand the requirement that the image of P in $C(k_v)$ is contained in a specified closed and open ("clopen") subset of $C(k_v)$. If v is an infinite place, this just means that we require P to be on some specified connected component(s) of $C(k_v)$; for finite places, we can take something like a "residue class". With this notion, the Main Conjecture 9.1 above is equivalent to the following statement.

Let C/k be a curve as above. Specify local conditions at finitely many places of k and assume that there is no point in $C(k)$ satisfying these conditions. Then there is some $n \geq 1$ such that no point in $\prod_v X_v \subset C(\mathbb{A}_k)_\bullet$ survives the n-covering of C, where X_v is the set specified by the local condition at those places where a condition is specified, and $X_v = C(k_v)$ (or $\pi_0(C(k_v))$) otherwise.

This says that the "finite abelian" obstruction (equivalently, the Brauer-Manin obstruction) is the only obstruction against weak approximation in $C(\mathbb{A}_k)_\bullet$.

For the weak form of the conjecture, we replace the n-covering by an arbitrary covering.

We see that the conjecture implies that we can decide if a given finite collection of local conditions can be satisfied by a rational point. Now the question is how practical it might be to actually do this in concrete cases. For certain classes of curves and specific values of n, it may be possible to explicitly and efficiently find the relevant twists. For example, this can be done for hyperelliptic curves and $n = 2$, compare [BS2]. However, for general curves and/or general n, this approach is likely to be infeasible.

On the other hand, assume that we can find $J(k)$ explicitly, where J, as usual, is the Jacobian of C. This is the case (at least in principle) when $\text{Ш}(k, J)_{\text{div}} = 0$. Then we can approximate $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ more and more precisely by looking at the images of $C(\mathbb{A}_k)_\bullet$ and of $J(k)$ in $\prod_{v \in S} J(k_v)/NJ(k_v)$ for increasing N and finite sets S of places of k. If $C(k)$ is empty and the Main Conjecture holds, then for some choice of S and N, the two images will not intersect, giving an explicit proof that $C(k) = \emptyset$. An approach like this was proposed (and carried out for some twists of the Fermat quartic) by Scharaschkin [Sc]. See [Fl] for an implementation of this method and [BS3] for improvements. In [PSS], this procedure is used to rule out rational points satisfying certain local conditions on a genus 3 curve whose Jacobian has Mordell-Weil rank 3.

In order to test the conjecture, Nils Bruin and the author conducted an experiment, see [BS1]. We considered all genus 2 curves over $\mathbb{Q}$ of the form

$$y^2 = f_6\, x^6 + f_5\, x^5 + \cdots + f_1\, x + f_0 \tag{9.1}$$

with coefficients $f_0, \ldots, f_6 \in \{-3, -2, \ldots, 3\}$. For each isomorphism class of curves thus obtained, we attempted to decide if there are rational points or not. On about 140 000 of these roughly 200 000 curves (up to isomorphism), we found a (fairly) small rational point. Of the remaining about 60 000, about half failed to have local points at some place. On the remaining about 30 000 curves, we performed a 2-descent and found that for all but 1 492 curves C, $C(\mathbb{A}_\mathbb{Q})_\bullet^{\text{2-ab}} = \emptyset$, proving that $C(\mathbb{Q}) = \emptyset$ as well. For the 1 492 curves that were left over, we found generators of the Mordell-Weil group (assuming the Birch and Swinnerton-Dyer Conjecture for a small number of them) and then did a computation along the lines sketched above. This turned out to be successful for *all* curves, proving that none of them has a rational point. The conclusion is that the Main Conjecture holds for curves C as in (9.1) if $C(\mathbb{Q}) = \emptyset$, assuming $Ш(\mathbb{Q}, J)_{\rm div} = 0$ for the Jacobian J if C is one of the 1 492 curves mentioned, and assuming in addition the Birch and Swinnerton-Dyer Conjecture if C is one of 42 specific curves.

At least in case $C(k)$ is empty, there are heuristic arguments due to Poonen [Po2] that suggest that an even stronger form of our conjecture might be true.

Conjecture 9.4 (Poonen)**.** *Let C be a smooth projective geometrically connected curve of genus ≥ 2 over a number field k, and assume that $C(k) = \emptyset$. Assume further that C has a rational divisor class of degree 1, and let $\iota : C \to J$ be the induced embedding of C into its Jacobian J. Then there is a finite set S of finite places of good reduction for C such that the image of $J(k)$ in $\prod_{v\in S} J(\mathbb{F}_v)$ does not meet $\prod_{v\in S} \iota(C(\mathbb{F}_v))$.*

Note that under the assumption $Ш(k, J)_{\rm div} = 0$, we must have a rational divisor (class) of degree 1 on C whenever $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} \neq \emptyset$, compare Cor. 6.6, so the condition above is not an essential restriction.

In the following, we would like to discuss how an effective version of Mordell's Conjecture could be obtained under some (as we think) plausible assumptions like the Strong Chabauty and Eventually Small Rank conjectures given below. All known proofs of Mordell's Conjecture are non-effective. On the other hand, Elkies [El] has shown that 'effective Mordell' is implied by 'effective ABC', but a proof of the ABC Conjecture (let alone an effective one) seems to be far away.

Assuming $Ш(k, J)_{\rm div} = 0$, we know that Conj. 9.1 holds when $J(k)$ is finite. We would like to extend this at least to the case when $J(k)$ has small rank. Recall that Chabauty's method allows us to find an explicit upper bound on the number of rational points on C when the rank r of $J(k)$ is less than the genus g of C. This is done by intersecting the closure of $J(k)$ in $J(k_v)$ with the image of $C(k_v)$, for a suitable place v of k. See for example [Ch], [Co] or [Sto]. Under the rank condition, this intersection is finite. If the difference $g - r$ is at least 2, then the system of equations one derives is overdetermined, and so one expects not to find extraneous solutions. This has been noted by a number of people working on Chabauty techniques, although a precise statement along the lines of the formulation below may not have appeared in the literature. We capture our expectation in the following conjecture, which refers to a slightly more general situation.

Conjecture 9.5 (Strong Chabauty Conjecture)**.** *Let C/k be a curve as above and assume that there is a nonconstant morphism $\iota : C \to A$ into an abelian variety* *such that $\iota(C)$ generates A (hence A is a factor of the Jacobian of C). Assume that the rank r of $A(k)$ is at most* $\dim A - 2$. *Then there is a finite subscheme $Z \subset A$ and a set of places v of k of density 1 such that the topological closure of $A(k)$ in $A(k_v)$ meets $\iota(C)$ in a subset of $Z(k_v)$.*

It should be possible to collect empirical evidence for this conjecture, for example by studying a number of hyperelliptic genus 3 curves whose Jacobians have Mordell-Weil rank 1.

Theorem 9.6. *Let C/k be a curve such that Conj. 9.5 holds for C. Assume that the rank condition $r \le \dim A - 2$ is satisfied and that $Ш(k, A)_{\mathrm{div}} = 0$. Then C is excellent w.r.t. abelian coverings.*

Proof: If $P \in C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, we have $\iota(P) \in A(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \overline{A(k)}$, and so $\iota(P_v) \in Z(k_v)$ for the set of places v of density 1 whose existence the conjecture asserts. By Thm. 3.11, we find that $\iota(P) \in Z(k)$. Finally, use Thm. 8.2 for $\iota^{-1}(Z(k))$. □

Now one does not expect the ranks of Mordell-Weil groups to be very large, even for abelian varieties of high dimension. So it is very likely that the Jacobians of high-degree coverings of a curve C will have (factors of) Mordell-Weil ranks that are small compared to their dimension.

Conjecture 9.7 (Eventually Small Rank Conjecture)**.** *Let C/k be a curve as above, embedded via ι in its Jacobian J. Then there is some $n \ge 1$ such that for all twists $D \to C$ of the pull-back of $J \xrightarrow{\cdot n} J$ to C such that D has points everywhere locally, there is a factor A of the Jacobian of D such that the Mordell-Weil rank of A is at most* $\dim A - 2$.

Of course, in conjuction with Chabauty's method, this conjecture would also provide another proof of Mordell's Conjecture. In this context, it has been around as a 'folklore' statement for a while (with $A = \mathrm{Jac}(D)$, and $g(D) - 1$ replacing $\dim A - 2$).

Now let us put together the last two conjectures.

Theorem 9.8. *Assume that*

(1) $Ш(k, A)_{\mathrm{div}} = 0$ *for all abelian varieties A/k;*
(2) *the Strong Chabauty Conjecture 9.5 is valid;*
(3) *the Eventually Small Rank Conjecture 9.7 is valid.*

Then all curves C/k of genus at least 2 are excellent w.r.t. solvable coverings. In particular, all curves over k are good.

Proof: There is nothing to show when C has genus 0, and the first assumption implies that C is (very) good when the genus is 1. So we can assume that C has genus at least 2. If C does not have a rational divisor class of degree 1, then by the first assumption again, $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \emptyset$, and there is nothing to prove. Otherwise, we can embed C into its Jacobian J. If J has a factor A satisfying the rank condition $r \le \dim A - 2$, then under the first two assumptions, C is even excellent w.r.t. abelian coverings by Thm. 9.6.

If there is no such factor of J, then by Conj. 9.7, there is some $n \ge 2$ such that for every twist $D \to C$ of the n-covering of C with points everywhere locally, the Jacobian of D has a factor satisfying the rank condition. By the discussion above, D is excellent w.r.t. abelian coverings. Prop. 8.4 now implies that C is excellent w.r.t. solvable coverings. □

Unfortunately, we cannot conclude in this way that C is excellent w.r.t. abelian coverings.

Assuming the Strong Chabauty Conjecture and triviality of $Ш(k, A)_{\text{div}}$, we can explicitly determine the set of rational points on any curve C/k that maps non-constantly into an abelian variety A such that the rank of $A(k)$ is at most $\dim A - 2$. This is done by applying Chabauty's method to the image C' of C in A: for all places v in a set of density 1, the points in the intersection of $C'(k_v)$ and the closure of $A(k)$ in $A(k_v)$ will be algebraic and can therefore be identified. In particular, we can check whether they are rational or not. This determines $C'(k)$, from which it is easy to find $C(k)$.

If we also assume the Eventually Small Rank Conjecture as in Thm. 9.8 above, then there will be some n such that we can determine $D_\xi(k)$ for all twists D_ξ of the n-covering of C such that $D_\xi(\mathbb{A}_k)_\bullet \neq \emptyset$. By the Descent Theorem 5.1, this means that we can find $C(k)$ as well. This proves the following result.

Theorem 9.9. *Under the assumptions of Theorem 9.8, there is an algorithm that determines the set of rational points $C(k)$ on any curve C over k of genus $g \ge 2$.*

We can try to get by without the Strong Chabauty Conjecture. Let us say that a curve C/k (of genus ≥ 2) has the *eventually rank zero (ERZ) property* if there is some $n \ge 1$ such that the Jacobian of every twist of the n-covering of C that has points everywhere locally has a factor A such that $A(k)$ is finite and $Ш(k, A)_{\text{div}}$ is trivial. Then by Cor. 8.6 and the discussion above, the following is clear.

Theorem 9.10. *Let C/k be a smooth projective geometrically connected curve of genus $g \ge 2$. If C has the ERZ property, then C is excellent w.r.t. solvable coverings, and we can determine $C(k)$ algorithmically.*

Now how likely is it for a curve to have the ERZ property? It is widely believed that a positive fraction (maybe even half) of all simple abelian varieties have rank zero. If we assume this, then there is $0 \le q < 1$ such that a "random" abelian variety over k has no factor of rank zero with probability $\le q$. Now assume that the abelian varieties occurring as factors of the Jacobians of twists of the n-covering of C are random in a suitable sense. Since each such Jacobian has at least $\tau(n)$ factors, where $\tau(n)$ is the number of divisors of n, we find that the expected number of twists failing the rank zero condition is

$$E(n) \le q^{\tau(n)} \# \operatorname{Sel}^{(n)}(k, C) \,,$$

where $\operatorname{Sel}^{(n)}(k, C)$ is the n-Selmer set of C, which parametrizes the twists of the n-covering of C with points everywhere locally. Let J be the Jacobian of C. If we assume that $Ш(k, J)_{\text{div}} = 0$ and take $n = 6^{N-1}$ (for some $N \ge 1$), then

$$\# \operatorname{Sel}^{(n)}(k, C) \le \# \operatorname{Sel}^{(n)}(k, J) \le c\, n^r$$

with a constant c, where r is the rank of $J(k)$. We also have $\tau(n) = N^2$. This implies that $E(n) \le c\, q^{N^2} 6^{r(N-1)}$ tends to zero as N becomes large. So if we assume $Ш(k, J)_{\text{div}} = 0$ and a positive probability for rank zero, then "almost all" curves should have the ERZ property.

Relation to the Section Conjecture. There is some relation of our Main Conjecture 9.1 and its weak form 9.3 to Grothendieck's anabelian "Hauptvermutung" (see his letter to Faltings [Gr]). I am grateful to Jordan Ellenberg for pointing me to this connection and to Florian Pop for providing some expert information.

Let X be a smooth projective geometrically connected k-variety. We will denote by $\bar{X}$ its base-change to $\bar{k}$. We fix an algebraic closure $\overline{k(X)}$ of the function field $k(X)$. If K is a field, we write $\mathcal{G}_K$ for the absolute Galois group $\operatorname{Gal}(\bar{K}/K)$. We let $\pi_1(X)$ ($\pi_1(\bar{X})$) denote the Galois group of the maximal subextension of $k(X)$ ($\bar{k}(X)$) that is unramified at all points of X ($\bar{X}$). $\pi_1(X)$ is called the *arithmetic fundamental group* and $\pi_1(\bar{X})$ the *geometric fundamental group* of X. They classify the connected finite étale coverings of X (resp., $\bar{X}$). We let $\Pi(X)$ denote the quotient of $\pi_1(X)$ by the kernel of the maximal abelian quotient map $\pi_1(\bar{X}) \to \pi_1(\bar{X})^{\text{ab}}$ (which is a closed normal subgroup of $\pi_1(X)$); then we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G}_{\bar{k}(X)} & \longrightarrow & \mathcal{G}_{k(X)} & \longrightarrow & \mathcal{G}_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \pi_1(\bar{X}) & \longrightarrow & \pi_1(X) & \longrightarrow & \mathcal{G}_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \pi_1(\bar{X})^{\text{ab}} & \longrightarrow & \Pi(X) & \longrightarrow & \mathcal{G}_k & \longrightarrow & 0
\end{array}$$

A rational point $P \in X(k)$ induces a section $\sigma_P : \mathcal{G}_k \to \pi_1(X)$ (well-defined up to conjugation by an element of $\pi_1(\bar{X})$). The "Section Conjecture", which is a special case of Grothendieck's "Hauptvermutung" states that (e.g.) if X is a smooth projective curve of genus $g \geq 2$, then every section arises from a rational point. We say that X has the *section property* if this statement holds for X, and we say that X has the *weak section property* if $X(k) = \emptyset$ implies that there is no section as above. We can also consider the "birational version." A "birational" section $\mathcal{G}_k \to \mathcal{G}_{k(X)}$ is said to be *geometric* if its image is contained in the decomposition group of a rational point on X. We say that X has the *birational section property* if all such sections are geometric, and X was the *weak birational section property* if $X(k) = \emptyset$ implies that there is no birational section. See Koenigsmann's paper [Koe].

We now want to relate these notions to finite étale coverings. Sections as above correspond to projective systems of such coverings, so we will first define these objects.

Definition 9.11. A *pro-covering* of X is a projective system of geometrically connected coverings $(Y_i, G_i) \in \mathcal{C}ov(X)$ such that every connected finite étale covering $\bar{Y}$ of $\bar{X}$ is a quotient of some $\bar{Y}_i$.

An *abelian pro-covering* of X is a projective system of geometrically connected coverings $(Y_i, G_i) \in \mathcal{A}b(X)$ such that every connected abelian finite étale covering $\bar{Y}$ of $\bar{X}$ is a quotient of some $\bar{Y}_i$.

It is clear what an isomorphism of pro-coverings or of abelian pro-coverings of X should be. We will denote the set of isomorphism classes of pro-coverings (abelian pro-coverings) of X by $\widehat{\mathcal{C}ov}(X)$ ($\widehat{\mathcal{A}b}(X)$) and use the notation $(\hat{Y}, \hat{G})$ for elements of these sets. $\hat{Y}$ can be considered as a "pro-variety" over k and $\hat{G}$ as a pro-finite group with a continuous action of $\mathcal{G}_k$. As a pro-finite group, $\hat{G}$ is isomorphic to $\pi_1(\bar{X})$ ($\pi_1(\bar{X})^{\mathrm{ab}}$), and this isomorphism is uniquely determined up to an inner automorphism of the latter group.

If $k \subset K$ is any field extension, we denote by $\hat{Y}(K)$ the set of all compatible sequences of points in the $Y_i(K)$; this does not depend on the system (Y_i) we choose to represent $\hat{Y}$. In a similar way, we define $\hat{Y}(\mathbb{A}_k)_\bullet$.

We then have the following correspondences.

Proposition 9.12. *There is a canoncial bijection between $\widehat{\mathcal{C}ov}(X)$ and the set of conjugacy classes of sections to $\pi_1(X) \to \mathcal{G}_k$.*

Similarly, there is a canonical bijection between $\widehat{\mathcal{A}b}(X)$ and the set of conjugacy classes of sections to $\Pi(X) \to \mathcal{G}_k$.

PROOF: We will prove the first statement; the proof in the abelian case is analogous. Let L be the maximal subfield of $\overline{k(X)}$ that is unramified at all points of X, so that $\pi_1(X) = \mathrm{Gal}(L/k(X))$.

It is clear that conjugacy classes of sections are in canonical bijection with isomorphism classes of diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(\bar{X}) & \longrightarrow & \pi_1(\bar{X}) \rtimes \mathcal{G}_k & \longrightarrow & \mathcal{G}_k & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & \pi_1(\bar{X}) & \longrightarrow & \pi_1(X) & \longrightarrow & \mathcal{G}_k & \longrightarrow & 0 \end{array}$$

where two such diagrams are considered isomorphic when the induced automorphism of the semidirect product $\pi_1(\bar{X}) \rtimes \mathcal{G}_k$ is conjugation by an element of $\pi_1(\bar{X})$. On the other hand, such a diagram corresponds to a subfield $K \subset L$ such that k is algebraically closed in K and $K\bar{k} = L$, and isomorphic diagrams correspond to subfields conjugate under $\pi_1(\bar{X}) = \mathrm{Gal}(L/\bar{k}(X))$. Finally, every pro-covering $\hat{Y}$ of X gives rise to such a subfield $K = \varinjlim k(Y_i)$, and conversely, a subfield K as above gives rise to a pro-covering (since K is a filtered union of finite extensions $K_i/k(X)$ such that $K_i\bar{k}/\bar{k}(X)$ is Galois). Two pro-coverings are isomorphic if and only if the corresponding subfields of L are conjugate under $\pi_1(\bar{X})$. □

In these terms, the Section Conjecture states that every $\hat{Y} \in \widehat{\mathcal{C}ov}(X)$ has a k-rational point.

Lemma 9.13. *A point in $X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ (in $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$) gives rise to at least one element in $\widehat{\mathcal{C}ov}(X)$ (in $\widehat{\mathcal{A}b}(X)$). When X is a curve of genus $g \geq 2$, any such element comes from at most one adelic point on X.*

Proof: For a given point P in $X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ (in $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$), consider the set of all geometrically connected finite étale (abelian) coverings of X that lift P. Since there are only finitely many twists of every given covering that lift P, there will be at least one projective system that covers all finite étale coverings of $\bar{X}$.

For the second statement, it suffices to consider the abelian variant. So assume that $P, Q \in X(\mathbb{A}_k)_\bullet$ are both in the image of $\hat{Y}(\mathbb{A}_k)_\bullet$. This means that their difference in $J(\mathbb{A}_k)_\bullet$, where J is the Jacobian of X, is divisible by every $n \geq 1$, since P and Q lift to the same n-covering of X. Since the divisible subgroup of $J(\mathbb{A}_k)_\bullet$ is trivial, we must have that $P = Q$. □

Let us write $\hat{\pi} : \hat{Y} \to X$ for the structure map, then the result of the lemma implies that

$$X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \bigcup_{\hat{Y} \in \widehat{\mathcal{C}ov}(X)} \hat{\pi}(\hat{Y}(\mathbb{A}_k)_\bullet) \quad \text{and} \quad X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \bigcup_{\hat{Y} \in \widehat{\mathcal{A}b}(X)} \hat{\pi}(\hat{Y}(\mathbb{A}_k)_\bullet)\,.$$

We can use this characterization to prove the following result on products of varieties.

Proposition 9.14. *Let X and Y be two smooth projective geometrically connected k-varieties. Then*

$$(X \times Y)(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \times Y(\mathbb{A}_k)_\bullet^{\text{f-cov}}$$

and

$$(X \times Y)(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \times Y(\mathbb{A}_k)_\bullet^{\text{f-ab}}.$$

PROOF: By Prop. 5.8, we have

$$(X \times Y)(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \times Y(\mathbb{A}_k)_\bullet^{\text{f-cov}}.$$

So let $P \in X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$, $Q \in Y(\mathbb{A}_k)_\bullet^{\text{f-cov}}$. Then there are pro-coverings $\hat{\pi} : \hat{W} \to X$ and $\hat{\rho} : \hat{Z} \to Y$ such that $\hat{\pi}(\hat{W}(\mathbb{A}_k)_\bullet) = \{P\}$ and $\hat{\rho}(\hat{Z}(\mathbb{A}_k)_\bullet) = \{Q\}$. But then $\hat{W} \times \hat{Z}$ is a pro-covering of $X \times Y$ (here we use that $\pi_1(\bar{X} \times \bar{Y}) = \pi_1(\bar{X}) \times \pi_1(\bar{Y})$), hence

$$\{(P, Q)\} = (\hat{\pi} \times \hat{\rho})((\hat{W} \times \hat{Z})(\mathbb{A}_k)_\bullet) \subset (X \times Y)(\mathbb{A}_k)_\bullet^{\text{f-cov}}.$$

The proof for the abelian variant is analogous (and would also carry over in the obvious way to the case of solvable coverings). □

Remark 9.15. The preceding result really belongs into Section 5. It would be interesting to know if there is a more direct proof. Note also that the assumption that X and Y are geometrically connected is inessential: by Prop. 5.16, we can assume that X and Y are connected. If X (say) is connected, but not geometrically connected, then $(X \times Y)(\mathbb{A}_k)_\bullet = X(\mathbb{A}_k)_\bullet = \emptyset$, and the statement is trivially true.

We now find that the Section Conjecture implies the weak form of our Main Conjecture 9.3.

Theorem 9.16. *Let C/k be a smooth projective geometrically connected curve of genus $g \geq 2$. If C/k has the section property, then C is excellent w.r.t. all coverings.*

PROOF: Let $P \in C(\mathbb{A}_k)_\bullet^{\text{f-cov}}$. By Lemma 9.13, there is $\hat{D} \in \widehat{\mathcal{C}ov}(C)$ such that P is the image of $\hat{D}(\mathbb{A}_k)_\bullet$ in $C(\mathbb{A}_k)_\bullet$. By the section property of C, $\hat{D}(k)$ is nonempty. Since $\hat{D}(k) \subset \hat{D}(\mathbb{A}_k)_\bullet$, P is the image of a rational point on $\hat{D}$ and therefore is itself a rational point. □

Now the analogue of the section property for the abelianized sequence

$$0 \longrightarrow \pi_1(\bar{X})^{\text{ab}} \longrightarrow \Pi(X) \longrightarrow \mathcal{G}_k \longrightarrow 0$$

will in general be false. For example, every rational point on Pic^1_C will give rise to a section (Pic^1_C has the same geometric π_1^{ab} as C), but $C(k)$ may well be empty even though $\text{Pic}^1_C(k)$ is not. So there is no similarly direct connection to the strong form of our Main Conjecture. However, our conjectures can be seen as a form of the Hasse Principle for sections or pro-coverings, as follows.

Proposition 9.17. *Conjecture 9.1 (its weak form 9.3) for a curve C/k of genus at least 2 is equivalent to the following statement.*

Every $\hat{D} \in \widehat{\mathcal{A}b}(C)$ ($\widehat{\mathcal{C}ov}(C)$) such that $\hat{D}(\mathbb{A}_k)_\bullet \neq \emptyset$ has a k-rational point.

PROOF: This is clear in view of the discussion above. □

Unfortunately not much is known about the section property in general. On the other hand, Koenigsmann [Koe] proves that curves of genus ≥ 2 have the birational section property over p-adic fields (and the reals). We can use this to show that the birational section property holds over number fields k in certain cases.

Theorem 9.18. *Let C/k be a smooth projective geometrically connected curve of genus $g \geq 2$ that is excellent w.r.t. all coverings. Then C has the birational section property.*

PROOF: We first show that C has the weak birational section property. So assume that we have a birational section $\sigma : \mathcal{G}_k \to \mathcal{G}_{k(C)}$. This induces birational sections $\sigma_v : \mathcal{G}_{k_v} \to \mathcal{G}_{k_v(C)}$ for all places v of k. By Prop. 2.4, (2) in [Koe], C has the birational section property over all k_v, so for each v, there is a point $P_v \in C(k_v)$ such that the image of σ_v is contained in the decomposition group of P_v. For the induced section $\bar{\sigma} : \mathcal{G}_k \to \pi_1(C)$, this means that the corresponding $\hat{D} \in \widehat{\mathcal{C}ov}(C)$ lifts each of the points P_v, hence $P = (P_v) \in C(\mathbb{A}_k)_\bullet$ is in fact in $C(\mathbb{A}_k)_\bullet^{\text{f-cov}}$. By assumption, $P \in C(k)$, so $C(k) \neq \emptyset$, hence C has the weak birational section property.

Now by Prop. 8.5, all curves D covering C are also excellent w.r.t. all coverings and, by what we have just shown, enjoy the weak birational section property. The proof of Lemma 1.7 in [Koe] then shows that this implies the birational section property for C. □

Corollary 9.19. *Under the assumptions of Thm. 9.8, all curves of genus ≥ 2 over k have the birational section property.*

Even without assuming all the conjectures in Thm. 9.8, we can prove the birational section property for a number of examples, at least over $\mathbb{Q}$:

(1) By the discussion in the preceding section, all modular curves $X_0(N)$, $X_1(N)$, $X(N)$ of genus at least two have the birational section property.
(2) All "small" (i.e., given by an integral equation $y^2 = f(x)$ with coefficients of absolute value at most 3) genus 2 curves without rational points have the birational section property (assuming $Ш(\mathbb{Q}, J)_{\text{div}} = 0$ for 1492 curves and in addition the Birch and Swinnerton-Dyer conjecture for 42 among them). This means that these curves do not admit birational sections.

As a final remark, we note that the weaker property

$$C(\mathbb{A}_k)_\bullet^{\text{f-cov}} \neq \emptyset \implies C(k) \neq \emptyset$$

(this is the weakest property in the "matrix" (6.1)), if satisfied by all finite étale coverings of a curve C of genus at least 2, already implies that C is excellent w.r.t. all coverings. Indeed, let $\hat{D} \in \widehat{\mathcal{C}ov}(C)$ be a pro-covering corresponding to a point $P \in C(\mathbb{A}_k)_\bullet^{\text{f-cov}}$, represented by a family (D_i, G_i) of torsors in $\mathcal{C}ov(C)$. By assumption, $D_i(k) \neq \emptyset$ for all i (since $D_i(\mathbb{A}_k)_\bullet^{\text{f-cov}} \neq \emptyset$, compare the proof of Prop. 5.13). Since $D_i(k)$ is finite for all i, there must be a compatible system of k-rational points on the D_i, which means that $\hat{D}(k) \neq \emptyset$. Since P is the image of $\hat{D}(\mathbb{A}_k)_\bullet \supset \hat{D}(k)$, P is already rational.

In particular, this shows that Poonen's Conjecture 9.4 (together with the assumption that $Ш(k, A)_{\text{div}} = 0$ for abelian varieties A over k, in order to deal with curves without embedding into their Jacobians) would imply that all curves of genus at least two are excellent w.r.t. all coverings. By Thm. 9.18 above, this would in turn imply the birational section property for all these curves.

One point here that is philosophically of interest is that we can (in principle, assuming $Ш_{\text{div}} = 0$) easily verify Poonen's Conjecture (or the property displayed above) for any given curve by a finite computation. In contrast, this is not so clear for the statement of our Main Conjecture.

Relation to ideas of Bogomolov and Tschinkel. Now we use some of the ideas of Bogomolov and Tschinkel on coverings, see [BT]. Let C, C' be curves over $\bar{k}$. Then we say that "C lies above C'" if there is another curve D, an étale morphism $D \to C$, and a dominant morphism $D \to C'$. (So C dominates C' up to étale coverings.) We write $C \Longrightarrow C'$. Without loss of generality, we can assume that $D \to C$ is Galois. Then for some finite extension K/k, $D \to C$ is defined over K and gives rise to a C-torsor in $\mathcal{C}ov(C_K)$.

Proposition 9.20. *Let C, C' be curves over k and assume that $C \Longrightarrow C'$ (over $\bar{k}$). If for all finite extensions K of k, C'_K is excellent w.r.t. all coverings, then we also have for all finite extensions K of k that C_K is excellent w.r.t. all coverings.*

PROOF: Let $C \leftarrow D \to C'$ be the diagram over $\bar{k}$ coming from the fact that C lies above C' (with $D \to C$ Galois). By Prop. 8.5, we have $D(\mathbb{A}_K)_\bullet^{\text{f-cov}} = D(K)$ for all sufficiently large K (say, such that the diagram is defined over K). Now this implies that we also have the same property for all twists of D, using Prop. 8.3 and the fact that twists become isomorphic over some finite extension. Then by Prop. 8.4, we have $C(\mathbb{A}_K)_\bullet^{\text{f-cov}} = C(K)$ for all sufficiently large K. But then by Prop. 8.3 again, this follows for all K. □

Corollary 9.21. *If the curve $C_6 : y^2 = x^6 + 1$ has the property that C_6/k is excellent w.r.t. all coverings for all number fields k, then all hyperelliptic curves over all number fields are excellent w.r.t. all coverings. The same statement holds with 'all coverings' replaced by 'solvable coverings'. Similarly for $C_5 : y^2 = x^5 + 1$ in place of C_6.*

PROOF: Bogomolov and Tschinkel show that every hyperelliptic curve lies above C_6 and C_5. Furthermore, the étale covering they construct in the proof is solvable, so that one can transfer the result of the preceding proposition to the solvable variant. □

In fact, the statement holds for a considerably larger class than hyperelliptic curves (for excellence w.r.t. all coverings, say), compare [Po1]. Bogomolov and Tschinkel conjecture that $C_6 \implies C$ for all curves C (Conjecture 1.1 in [BT]). For our purposes, the converse ($C \implies C_6$ for all C of genus at least 2) would be more useful, but it is perhaps not so clear whether this can really be expected to hold.

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