STABILITY OF PEAK SOLUTIONS OF A NON-LINEAR TRANSPORT EQUATION ON THE CIRCLE

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Abstract. We study solutions of a transport-diffusion equation on the circle. The velocity of turning is given by a non-local term that models attraction and repulsion between elongated particles. Having mentioned basics like invariances, instability criteria and non-existence of time-periodic solutions, we prove that the constant steady state is stable at large diffusion. We show that without diffusion localized initial distributions and attraction lead to formation of several peaks. For peak-like steady states two kinds of peak stability are analyzed: first spatially discretized with respect to the relative position of the peaks, then stability with respect to non-localized perturbations. We prove that more than two peaks may be stable up to translation and slight rearrangements of the peaks. Our fast numerical scheme which is based on the Fourier-transformed system allows to study the long-time behaviour of the equation. Numerical examples show backward bifurcation, mixed-mode solutions, peaks with unequal distances, coexistence of one-peak and two-peak solutions and peak formation in a case of purely repulsive interaction.

1. Introduction

In this paper we analyze pattern forming ability and pattern stability for a one-dimensional non-linear transport-diffusion equation on the circle. The distinguishing feature of this equation is the non-local turning velocity that is determined by interactions between particles in various orientations: velocity is given by a convolution term of an interaction rate V with the distribution function. In its general form, the equation also includes a diffusion term.

Our interest in this equation is three-fold: It has been used to model the formation of F-actin bundles and networks of the cytosceleton [14], [15]. Secondly, this partial differential equation can be derived formally from a general integro-differential equation on the circle, [16]. This integro-differential equation has also been used to model F-actin aggregation [6]. Finally, the corresponding equation *on the real line* is interesting both mathematically and from the modelling viewpoint, see e.g. [2] or [4] for references and applications.

Consequently, several facts and analytical methods for the equation on the circle have been established: Primi et.al. [16] prove existence of solutions and find conditions on V such that non-constant stationary solutions of the transport-diffusion equation exist for small enough diffusion. Mogilner et.al. [15] analyze stability of a single peak in a discrete setting without diffusion. Chayes and Panferov [3] analyze existence and bifurcations of non-trivial stationary solutions on d-dimensional tori by minimizing an appropriate 'free energy' functional.

The starting points of our interest were the claim — between the lines — of Primi et.al. [16] that a certain integral condition allows to decide wether one or two peaks will form and the question what 'mass selection' means. In Section 6, Example 6.3 we show that single and double peaks may exist simultaneously. Example 6.1 shows a mixed mode solution (and a nice backward bifurcation),

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i.e. there is no mass selection at higher diffusion. The mixed mode solution seems to converge to two peaks of equal height with decreasing diffusion, but the time that is needed for convergence to two peaks increases rapidly.

However, our main interest concerns the stability of several peaks. The method used by Primi et.al. [16] of constructing peak-like solutions gives no information on their stability, neither does the bifucation argument of Chayes and Panferov [3] away from the first bifurcation. Fellner's and Raoul's integration method [4] is not applicable to equations on the circle. At least Mogilner's et.al. 'peak ansatz' can be generalized to n peaks, see Section 5.1. The underlying notion of stability here is important: In the 'peak ansatz' only peak-like perturbations are considered; nothing can be said about perturbations with distributed masses.

For a related integro-differential equation on the circle Geigant [8] proves stability of a single peak with respect to perturbations that are measures with compact support. She linearizes the integro-differential equation near the peak and calculates the solution of the linearized equation and its limit explicitly. In section 5.2 we also linearize the transport equation near peak solutions but then we use a different method: calculating the moments of the linearized equation yields that the perturbation converges to 0 in the case of two opposite peaks as well as in the case that the two peaks have the critical distance θ_0 where $V(\theta_0) = 0$. For more than two peaks we can show with similar arguments that the number of peaks is stable but not their relative position, i.e., after a perturbation the relative distances (in general) are no longer equal. A technical difficulty is that solutions are invariant with respect to translations: therefore, 'stabilty' always means stability *up to translation*.

The outline of the paper is as follows:
In Section 2 we establish some basic facts on the transport-diffusion equation, like conservation of mass and symmetries, non-existence of time-periodic solutions, correspondence between solutions of higher periodicity for V and general solutions for its 'rolled-up' version V_n. Linearization near the constant stationary state provides conditions on the interaction rate V and on the smallness of the diffusion coefficient such that non-constant stationary states exist. We also discuss the corresponding equation on the real line and its relation with the equation on the circle. This leads on to statements about invariance of local supports, local masses and local barycenters for the eqation on the circle. Note that there is no reasonable global notion of first moment or barycenter; later on we present Example 5.2 that demonstrates this.
In Section 3 we show that the constant stationary solution is globally stable if diffusion is large enough compared to the transport term.
In Sections 4 and 5 diffusion is zero. In Section 4 we establish convergence to peak solutions for initial functions with sufficiently small (disjoint) support(s).
Section 5 is dedicated to the stability results for peak solutions already discussed above: namely first the peak ansatz in Section 5.1 which yields instablity conditions for peak solutions. Secondly, in Section 5.2 stability of peaks with respect to perturbations by differentiable measures is explored.
Section 6 contains two numerical algorithms and several instructive examples. A fast method to calculate solutions of the partial differential equation is based on the Fourier representation of (1), see section 6.1. The advantage in speed of that numerical method over methods based on discretization of space has been already used by Geigant and Stoll [9] for the integro-differential equation on the circle. In Section 6.2 we implement the method of Primi et.al. [16] to construct stationary solutions, which is an iteration scheme. Interestingly, there are several examples where the scheme does not converge[1]; so that method is not always applicable to construct stationary

[1]Of course these are examples where the assumptions of Primi et.al. do not hold.

Method / Steady state	spectral	local, linear	local, non-linear	global
constant	ev (3)		[3] D large	Th. 3.2 D large
single peak	ev (16), Th. 5.3 $V'(0)>0$	Th. 5.6 $V\|_{]0,\frac{1}{2}[}>0$	Th. 4.2 cs	
two peaks	ev (16), Th. 5.3 $V'(\frac{1}{2})>0$	Th. 5.8, Cor. 5.9 $\tilde{V}$ four zeros	Th. 4.7 $V\|_{]0,\theta_1[}>0, V\|_{]\theta_2,\frac{1}{2}[}<0$, cs	
$n\geq 3$ peaks	ev (16), Th. 5.3 $V'(\frac{k}{n})>0\,\forall k$	Cor. 5.7, Th. 5.10 $V_n\|_{]0,\frac{1}{n}[}>0$	Cor. 4.4, Cor. 4.5 $V\|_{]0,\theta_1[}>0$, cs pert. n-per. or $V\|_{[\theta_1,\frac{1}{2}]}=0$	

TABLE 1. List of stability results, methods and assumptions. (ev = eigenvalues, cs = sufficiently small compact support, pert = perturbation, per = periodic)

solutions.
The first of our examples in Section 6.3 shows the simultaneous bifurcation of first and second mode. The second example shows a stable two-peaks like solution where the two peaks are not opposite. In the third example *stable* one-peak and two-peaks like solutions coexist at the same parameter values. In the last example we show that pattern formation may occur even if V is nowhere attracting.
In the discussion in Section 7 we detail similarities and differences between the transport-diffusion equation and a related integro-differential equation on the circle.

Table 1 lists steady states and methods to analyze stability. Skip the table on first reading and return to it as a reference list! To help remember the assumptions only keywords are given.

2. The non-linear transport equation with diffusion

Let $S^1=\mathbb{R}/\mathbb{Z}$ be a circle of length 1. If we denote by $p:\mathbb{R}\to S^1=\mathbb{R}/\mathbb{Z}$ the canonical projection, then we have associated maps p^* from functions on S^1 to functions on $\mathbb{R}$, where $p^*(f)=f\circ p$ is the associated 1-periodic function on $\mathbb{R}$, and p_* from (sufficiently fast decaying) functions on $\mathbb{R}$ to functions on S^1, where

$$p_*(g)(\theta)=\sum_{x\in\mathbb{R},p(x)=\theta} g(x)\,.$$

Definition 2.1. A *closed interval* I on S^1 is a closed connected subset that is not all of S^1. Then $I=p(I')$ for some closed interval $I'=[a,b]\subset\mathbb{R}$ (such that $b-a<1$), and we write $I=[p(a),p(b)]$ and call $\alpha=p(a)$ the *lower end* and $\beta=p(b)$ the *upper end* of I. If h is a function on S^1, we write

$$\int_a^b h(\theta)\,d\theta=\int_\alpha^\beta h(\theta)\,d\theta=\int_{[\alpha,\beta]} h(\theta)\,d\theta=\int_a^b p^*(h)(x)\,dx.$$

If $\theta,\psi\in I$, we write $\theta-\psi\in\mathbb{R}$ for the difference $\theta'-\psi'$ where $\theta',\psi'\in I'$ are such that $p(\theta')=\theta$, $p(\psi')=\psi$.

If $V:S^1\to\mathbb{R}$ is a function and $I=\,]a,b[\,\subset\mathbb{R}$ is an interval such that $p(I)\neq S^1$, we will (for simplicity) say that '$V>0$ on $]a,b[$' if $V>0$ on $p(I)$ (equivalently, $p^*(V)>0$ on I); similarly for half-open or closed intervals. In the same way, we write $V(a)$ for $V(p(a))$ if $a\in\mathbb{R}$.

2.1. **The equation on the circle.**

We want to model a process that describes the change of orientation of filaments over time. The orientation is given by an 'angle' $\theta \in S^1$. The density of filaments at time $t \geq 0$ with orientation $\theta \in S^1$ is given by $f(t, \theta)$. The filaments turn continuously; the velocity of turning is determined by interactions with other filaments on the circle. At the same time there is random reorientation. This kind of dynamics is described by the following transport equation with diffusion, which is also known as the *McKean-Vlasov Equation*:

$$\frac{\partial f}{\partial t}(t,\theta) = D\frac{\partial^2 f}{\partial \theta^2}(t,\theta) + \frac{\partial}{\partial \theta}\big((V * f(t,\cdot)) \cdot f(t,\cdot)\big)(\theta), \tag{1}$$

where $D \geq 0$ is the diffusion coefficient and $(V * f)(\theta) = \int_{S^1} V(\theta - \psi) f(\psi)\, d\psi$ is the convolution of V with f and gives the negative velocity of turning of filaments with orientation θ.

We assume that the interaction function $V : S^1 \to \mathbb{R}$ is *odd*, because interactions with filaments on opposite sides of θ must have similar consequences. In particular, $V(0) = 0$, i.e., there is no repulsion or attraction of filaments with the same orientation, and $V(\frac{1}{2}) = 0$, i.e., there is no interaction with filaments of opposite orientation. The sign of $V(\theta)$ is important. If $V(\theta) > 0$ for some interaction angle $0 < \theta < \frac{1}{2}$ then the two filaments move towards each other, we call this 'attracting'; if on the other hand $V(\theta) < 0$ for some $0 < \theta < \frac{1}{2}$ then the distance between the filaments becomes greater, they are 'repelling each other'. For odd $V \in \mathcal{C}^\infty(S^1)$ Primi et al. [16] prove a-priori estimates by which unique existence of smooth solutions of equation (1) can be shown. In Carrillo et al. [1] a well-posedness theory for weak measure solutions is developed.

The following easy statement will be useful.

Lemma 2.2. *Let $V, f \in \mathcal{C}(S^1)$ with V odd. Then*

$$\int_{S^1} (V * f)(\theta) f(\theta)\, d\theta = 0.$$

Proof. We have

$$\int_{S^1} (V * f)(\theta) f(\theta)\, d\theta = \int_{S^1}\int_{S^1} V(\theta - \psi) f(\psi)\, d\psi\, f(\theta)\, d\theta = \int_{S^1}\int_{S^1} V(\theta - \psi) f(\psi) f(\theta)\, d\psi\, d\theta.$$

If we swap ψ and θ in the last integral, it changes sign (since V is odd); therefore it must be zero. □

The following proposition states some basic facts on equation (1).

Proposition 2.3. *Equation (1) preserves mass, non-negativity, axial symmetry with respect to any axis and periodicity of initial functions. Moreover, the solution space is invariant under the group $O(2)$ of translations and reflections on S^1.*

Proof. Preservation of mass and positivity are shown by Primi et al. [16]. The remaining statements follow from the observation that the operator on the right hand side of equation (1) is $O(2)$-equivariant (for the reflections in $O(2)$, this uses that V is odd). □

Since the partial differential equation (1) lives on S^1, the equation turns into a discrete system of ODEs when it is Fourier transformed. We denote by

$$f_k = \int_{S^1} f(\theta) e^{-2\pi i k \theta}\, d\theta \quad \text{for } k \in \mathbb{Z}$$

the k-th Fourier coefficient of $f : S^1 \to \mathbb{R}$. Since f is real, $f_k = \bar{f}_{-k}$; if f is even or odd, then $f_k \in \mathbb{R}$ or $f_k \in i\mathbb{R}$, respectively. For differentiable functions one has $(f')_k = 2\pi i k f_k$, the Fourier coefficients of a convolution are $(V * f)_k = V_k f_k$, and the Fourier transform of a product is the convolution of the Fourier series, $(f \cdot g)_k = \sum_{l\in\mathbb{Z}} f_l g_{k-l}$.

Hence the Fourier transform of the transport-diffusion equation (1) is

$$\begin{aligned} \dot{f}_k(t) &= -(2\pi k)^2 D f_k + 2\pi i k \sum_{l\in Z} V_l f_l f_{k-l} \\ &= c_k\, f_k + 4\pi k \sum_{l\in\mathbb{Z}\setminus\{0,k\}} v_l f_l f_{k-l} \quad \text{for } k\in\mathbb{Z}, \end{aligned} \tag{2}$$

where the eigenvalues $c_k \in \mathbb{R}$ of the system and the $v_k \in \mathbb{R}$ are defined as

$$c_k = -(2\pi k)^2 D + 4\pi k f_0 v_k \quad \text{and} \quad v_k = \int_0^{\frac{1}{2}} V(\theta)\sin(2\pi k\theta)\, d\theta = \tfrac{i}{2} V_k. \tag{3}$$

Mass conservation is reflected by the equation $\dot{f}_0 = 0$. Using $f_{-k} = \bar{f}_k$, the equations with $k < 0$ are redundant. We have $c_k = c_{-k} = \bar{c}_k \in \mathbb{R}$ because $v_k = -v_{-k}$. By scaling D and V one may assume that the mass is 1, which we will do from now on:

$$f_0 = \int_{S^1} f(t,\theta)\, d\theta = 1 \quad \text{for all} \quad t \geq 0.$$

2.2. **No time-periodic solutions and bounds for stationary solutions.**

Chayes and Panferov [3] (see also the literature cited there) proved that there are no time-periodic solutions.[2] We subsume their results and arguments here for the 1-dimenional case. They define

$$W(\theta) = \int_0^\theta V(\psi)\, d\psi\,;$$

this makes sense as a function on S^1, since $\int_{S^1} V(\psi)\, d\psi = 0$. Note that W is an even function. The 'free energy' functional of (1) is defined as

$$\mathcal{E}(f) := D\int_{S^1} f(\theta)\ln(f(\theta))\, d\theta + \tfrac{1}{2}\int_{S^1}\int_{S^1} W(\theta-\psi) f(\theta) f(\psi)\, d\psi d\theta.$$

Proposition 2.4. *Let $D > 0$ and $f(t,\theta) > 0$ be a solution of equation (1).*
Then $\frac{d\mathcal{E}(f(t,.))}{dt} \leq 0$, with equality if and only if f is a stationary solution.
Any time-periodic solution $f(t,\theta) > 0$ of (1) is in fact a stationary solution.
If $D = 0$ it is sufficient to assume $f \geq 0$ in both statements.

Proof. Chayes and Panferov show that $\frac{d}{dt}\mathcal{E}(f(t,.)) \leq 0$ and that $\frac{d\mathcal{E}(f(t,.))}{dt}(t_0) = 0$ holds for non-negative f if and only if $Df'(t,\cdot) + (V * f(t_0,\cdot))f(t_0,\cdot) = 0$. These two facts imply that any time-periodic solution is indeed stationary. □

The following proposition collects some results on stationary solutions. The following ordinary differential equation and bounds for stationary solutions for the McKean-Vlasov equation on tori in dimension $d \geq 1$ have already been found by Chayes and Panferov [3]. We omit a proof which can be based on integration and estimation of the ODE.

[2] Chayes and Panferov [3] assume throughout their paper that the interaction potential W is of finite range. However, for their statements which we cite here and in the following this assumption is not used in the proofs.

Proposition 2.5 (Estimates for stationary solutions)**.** *Assume that $f \geq 0$ is a stationary solution of (1) with mass 1 and $D > 0$. Then f satisfies the following ordinary differential equation on S^1:*

$$D\frac{df}{d\theta} = -(V * f) \cdot f. \tag{4}$$

For $\theta_1, \theta_2 \in S^1$ we have

$$e^{-C|\theta_1-\theta_2|} \leq \frac{f(\theta_2)}{f(\theta_1)} \leq e^{C|\theta_1-\theta_2|} \quad \textit{where} \quad C := \max V/D.$$

In particular, $\frac{\max f}{\min f} \leq e^{C/2}$ and therefore

$$\max f \leq e^{C/2} \min f \leq e^{C/2} \quad \textit{and} \quad \min f \geq e^{-C/2} \max f \geq e^{-C/2}.$$

In any maximum $\theta_{\max}$ of a stationary solution f we have

$$\frac{d^2 f}{d\theta^2}(\theta_{\max}) \geq -\frac{\max V'}{D} f(\theta_{\max});$$

in any minimum $\theta_{\min}$ of a stationary solution f we have

$$\frac{d^2 f}{d\theta^2}(\theta_{\min}) \leq -\frac{\min V'}{D} f(\theta_{\min}).$$

Remarks:
i) Primi et al. [16] use the ODE (4) to set up an iterative procedure for approximating stationary solutions. See Section 6.2 below.
ii) If the diffusion coefficient D is large compared to V, then any stationary solution is near the constant solution (or only the constant solution exists).
iii) The inequalities for $\max f$ have a large right hand side when D becomes small. That leads us to expect that with decreasing D solutions may become large and maxima may be sharp peaks (the curvature is large).
iv) If the minimum of f is small, then it is wide (the curvature is small).
v) If $D = 0$ in equation (1), then f is a stationary solution if and only if $(V * f) \cdot f = 0$.

We state another simple consequence of equation (4).

Proposition 2.6. *Assume $D > 0$ in equation (1). If V is $\frac{1}{n}$-periodic and odd, then any stationary solution of (1) must also be $\frac{1}{n}$-periodic.*

Proof. $V(\theta + \frac{1}{n}) = V(\theta)$ for all $\theta \in S^1$ implies

$$(V * f)(\theta + \tfrac{1}{n}) = \int_{S^1} V(\theta + \tfrac{1}{n} - \psi) f(\psi)\, d\psi = \int_{S^1} V(\theta - \psi) f(\psi)\, d\psi = (V * f)(\theta)$$

for functions f on S^1. If f is a stationary solution of (1), then it is a solution of the ODE (4). Since $D > 0$, it follows that $f'/f = -\frac{1}{D}(V * f)$ is $\frac{1}{n}$-periodic. This implies that $f(\theta + \frac{1}{n}) = \gamma f(\theta)$ with some constant γ, and since $f > 0$, we must have $\gamma > 0$. Since obviously $\gamma^n = 1$, we have $\gamma = 1$, and f is $\frac{1}{n}$-periodic. □

2.3. **Solutions with higher periodicity.**

Let $n \geq 1$ and $V : S^1 \to \mathbb{R}$ be odd and continuous. We are interested in $\frac{1}{n}$-periodic solutions of equation (1). To understand these, the following functions V_n and $\tilde{V}_n$ will be useful.

$$V_n(\theta) = \sum_{j=0}^{n-1} V\Big(\theta - \frac{j}{n}\Big), \qquad \tilde{V}_n(\theta) = \sum_{j=0}^{n-1} V\Big(\frac{\theta - j}{n}\Big).$$

V_n and $\tilde{V}_n$ are continuous and odd; V_n is $\frac{1}{n}$-periodic. In particular,

$$V_n\Big(\frac{k}{2n}\Big) = 0 \quad \text{for } 0 \le k < 2n.$$

The following result shows that instead of considering $\frac{1}{n}$-periodic solutions of equation (1), we can consider solutions of (1) without higher periodicity, when we modify the parameters D and V accordingly.

Proposition 2.7. *Let $n \ge 1$ and V be odd. Then there is a 1-to-1 correspondence between $\frac{1}{n}$-periodic solutions f of equation (1) and solutions $\tilde{f}$ of*

$$\frac{\partial \tilde{f}}{\partial t}(t,\theta) = n^2 D \frac{\partial^2 \tilde{f}}{\partial \theta^2}(t,\theta) + \frac{\partial}{\partial \theta}\big((\tilde{V}_n * \tilde{f}(t,\cdot))\tilde{f}(t,\cdot)\big)(\theta), \tag{5}$$

namely via $\tilde{f}(t, n\theta) = f(t,\theta)$.

Proof. Let $f(t,\cdot)$ be a $\frac{1}{n}$-periodic solution of (1). Define $\tilde{f}(t,\theta) = f(t,\frac{\theta}{n})$. Then $\tilde{f}$ has mass 1, and we have

$$\begin{aligned}(V * f(t,\cdot))\Big(\frac{\theta}{n}\Big) &= \sum_{j=0}^{n-1} \int_{\frac{j}{n}}^{\frac{j+1}{n}} V\Big(\frac{\theta}{n} - \psi\Big) f(t,\psi)\, d\psi = \sum_{j=0}^{n-1} \int_0^{\frac{1}{n}} V\Big(\frac{\theta}{n} - \Big(\psi + \frac{j}{n}\Big)\Big) f\Big(t, \psi + \frac{j}{n}\Big)\, d\psi \\ &= \int_0^{\frac{1}{n}} V_n\Big(\frac{\theta}{n} - \psi\Big) f(t,\psi)\, d\psi = \frac{1}{n}\int_0^1 V_n\Big(\frac{\theta-\psi}{n}\Big) f\Big(t,\frac{\psi}{n}\Big)\, d\psi = \frac{1}{n}(\tilde{V}_n * \tilde{f}(t,\cdot))(\theta).\end{aligned}$$

Also, $\frac{\partial \tilde{f}}{\partial \theta}(t,\theta) = \frac{1}{n}\frac{\partial f}{\partial \theta}(t,\frac{\theta}{n})$. Therefore,

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial t}(t,\theta) = \frac{\partial f}{\partial t}\Big(t,\frac{\theta}{n}\Big) &= D\frac{\partial^2 f}{\partial \theta^2}\Big(t,\frac{\theta}{n}\Big) + \frac{\partial}{\partial \theta}(V * f(t,\cdot) f(t,\cdot))\Big(\frac{\theta}{n}\Big) \\ &= n^2 D \frac{\partial^2 \tilde{f}}{\partial \theta^2}(t,\theta) + n\frac{\partial}{\partial \theta}\Big(\frac{1}{n}(\tilde{V}_n * \tilde{f}(t,\cdot))\tilde{f}(t,\cdot)\Big)(\theta) \\ &= n^2 D \frac{\partial^2 \tilde{f}}{\partial \theta^2}(t,\theta) + \frac{\partial}{\partial \theta}\big((\tilde{V}_n * \tilde{f}(t,\cdot))\tilde{f}(t,\cdot)\big)(\theta).\end{aligned}$$

The converse can be shown in the same way. □

This shows in particular that diffusion acts more strongly on solutions of higher periodicity. In fact, we have $\max \tilde{V}_n \le n \max V$, so the quotient $C = \max V/D$ in inequality 2.5 will be multiplied by a number $\le \frac{1}{n}$.

In terms of the Fourier transformed system (2), we have $f_k = 0$ for $n \nmid k$, $\tilde{f}_k = f_{nk}$, and $(\tilde{V}_n)_k = nV_{nk}$. So we only look at the equations with k a multiple of n and replace nk by k to obtain the system corresponding to equation (5).

2.4. **The equation on the real line.**

A similar PDE can also be considered with $\mathbb{R}$ instead of S^1 as the spatial domain,

$$\frac{\partial g}{\partial t}(t,x) = D\frac{\partial^2 g}{\partial x^2}(t,x) + \frac{\partial}{\partial x}\big((W * g(t,\cdot)) \cdot g(t,\cdot)\big)(x)\,. \tag{6}$$

Here $W : \mathbb{R} \to \mathbb{R}$ is odd and $g(t,\cdot)$ is assumed to decay sufficiently fast, so that the convolution $W * g(t,\cdot)$ is defined. Note that the convolution is here given by an integral over all of $\mathbb{R}$. There is the following relation between equations (1) and (6).

Proposition 2.8. *Let $V : S^1 \to \mathbb{R}$ be odd and define $W = p^*(V)$ (which is just V considered as a 1-periodic function on $\mathbb{R}$). Let $g : [0, T[\times \mathbb{R} \to \mathbb{R}$ be a solution of equation (6). Then $(t, \theta) \mapsto f(t, \theta) = p_*(g(t, \cdot))(\theta)$ is a solution of equation (1).*

Proof. We have

$$\begin{aligned}
\frac{\partial f}{\partial t}(t, \theta) &= \sum_{x:p(x)=\theta} \frac{\partial g}{\partial t}(t, x) \\
&= \sum_{x:p(x)=\theta} \Big(D \frac{\partial^2 g}{\partial x^2}(t, x) + \frac{\partial}{\partial x}\big((W * g(t, \cdot))(x) \cdot g(t, x)\big)\Big) \\
&= D \frac{\partial^2 f}{\partial \theta^2}(t, \theta) + \frac{\partial}{\partial \theta}\Big(\sum_{x:p(x)=\theta} \int_{-\infty}^{\infty} W(x - y) g(t, y)\, dy \cdot g(t, x)\Big) \\
&= D \frac{\partial^2 f}{\partial \theta^2}(t, \theta) + \frac{\partial}{\partial \theta}\Big(\int_{S^1} V(\theta - \psi) f(t, \psi)\, d\psi \cdot \sum_{x:p(x)=\theta} g(t, x)\Big) \\
&= D \frac{\partial^2 f}{\partial \theta^2}(t, \theta) + \frac{\partial}{\partial \theta}\big((V * f(t, \cdot)) f(t, \cdot)\big)(\theta)\,.
\end{aligned}$$

Here we use that

$$\int_{-\infty}^{\infty} p^*(V)(x - y) h(y)\, dy = \int_{S^1} V(p(x) - \psi) p_*(h)(\psi)\, d\psi\,. \qquad \square$$

The advantage of equation (6) over (1) is that it is easily shown to not only preserve mass, but also the first moment (or, equivalently, the barycenter) of $g(t, \cdot)$, whereas the notion of 'first moment' usually does not even make sense on S^1. (The following statements are surely not new, see e.g. Carrillo et.al. [2] or Raoul [17] if $D = 0$ on $\mathbb{R}^n$.)

Proposition 2.9. *Let g be a solution of (6).*

(1) *For any $a \in \mathbb{R}$, $(t, x) \mapsto g(t, x - a)$ is again a solution of (6).*
(2) *$(t, x) \mapsto g(t, -x)$ is again a solution of (6).*
(3) *$\int_{-\infty}^{\infty} g(t, x)\, dx$ is constant.*
(4) *$\int_{-\infty}^{\infty} x g(t, x)\, dx$ is constant.*

Proof. The first statement is clear (the operator on the right hand side is equivariant with respect to translations). Since W is assumed to be odd, the right hand side is also equivariant with respect to $x \mapsto -x$, which implies the second statement. For the third statement, we compute

$$\frac{d}{dt} \int_{-\infty}^{\infty} g(t, x)\, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\Big(D \frac{\partial g}{\partial x}(t, x) + (W * g(t, \cdot))(x) g(t, x)\Big)\, dx = 0\,,$$

using the decay properties of g. For the last statement, we have

$$\begin{aligned}
\frac{d}{dt} \int_{-\infty}^{\infty} x g(t, x)\, dx &= \int_{-\infty}^{\infty} x \frac{\partial}{\partial x}\Big(D \frac{\partial g}{\partial x}(t, x) + (W * g(t, \cdot))(x) g(t, x)\Big)\, dx \\
&= -\int_{-\infty}^{\infty} \Big(D \frac{\partial g}{\partial x}(t, x) + (W * g(t, \cdot))(x) g(t, x)\Big)\, dx \\
&= -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(x - y) g(t, y) g(t, x)\, dy\, dx \\
&= 0\,,
\end{aligned}$$

since W is odd, compare Lemma 2.2. □

Later, we will consider the case without diffusion (so with $D = 0$) in particular. In this situation, compact support is preserved, see Carrillo et.al. [2] for the nonlocal transport equation on $\mathbb{R}^n$. We prove this result here for completeness and with a different method.

Lemma 2.10. *Assume that W is bounded and that $D = 0$ in (6). Let $g \geq 0$ be a solution. If $g(0,\cdot)$ has support contained in $[a,b]$, then the support of $g(t,\cdot)$ is contained in $[a - Ct, b + Ct]$ for all $t > 0$, where $C = \|W\|_\infty \|g(0,\cdot)\|_1$.*

Proof. Let $g_+(t,x) = g(t, x + b + Ct)$. We show that $\operatorname{supp} g_+(t,\cdot) \subset \,]-\infty, 0]$. This implies that $\operatorname{supp} g(t,\cdot) \subset \,]-\infty, b + Ct]$. The argument for the lower bound is similar.

We have

$$\begin{aligned}\frac{\partial g_+}{\partial t}(t,x) &= \partial_t g(t, x+b+Ct) + C\partial_x g(t, x+b+Ct)\\ &= \partial_x\big((W * g(t,\cdot) + C)\cdot g(t,\cdot)\big)(x+b+Ct)\\ &= \partial_x\big((W + \|W\|_\infty) * g(t,\cdot)\cdot g(t,\cdot)\big)(x+b+Ct)\\ &= \partial_x\big((W + \|W\|_\infty) * g_+(t,\cdot)\cdot g_+(t,\cdot)\big)(x)\end{aligned}$$

Now

$$\begin{aligned}\frac{d}{dt}\int_0^\infty g_+(t,x)\,dx &= \int_0^\infty \partial_x\big((W + \|W\|_\infty) * g_+(t,\cdot)\cdot g_+(t,\cdot)\big)(x)\,dx\\ &= -\big((W + \|W\|_\infty) * g_+(t,\cdot)\cdot g_+(t,\cdot)\big)(0)\\ &= -\int_{-\infty}^\infty (W(-y) + \|W\|_\infty) g_+(t,y)\,dy\cdot g_+(t,0)\\ &\leq 0\,,\end{aligned}$$

since $g_+ \geq 0$ and $W + \|W\|_\infty \geq 0$. On the other hand,

$$\int_0^\infty g_+(0,x)\,dx = 0 \quad \text{and} \quad \int_0^\infty g_+(t,x)\,dx \geq 0\,,$$

so we must have $\int_0^\infty g_+(t,x)\,dx = 0$ for all t. □

If $W(x)$ is positive for positive x, then the solution converges to a single delta-peak. See also Fellner and Raoul [4, 5] and Raoul [17] for stationary solutions of the nonlocal transport equation on $\mathbb{R}^n$ and for their stability (and references given there). Fellner and Raoul transform the transport equation by considering the pseudo-inverse of the solution (so their method is completely different from ours).

Proposition 2.11. *Assume that W is continuously differentiable with $W'(0) > 0$, that $W(x) > 0$ for $x > 0$ and that $D = 0$ in (6). Let $g \geq 0$ be a solution such that $\operatorname{supp} g(0,\cdot) \subset [a,b]$. Then $g(t,\cdot)$ has support contained in $[a,b]$ for all $t > 0$, and it converges to a delta distribution $m\delta_c$ with $m = \|g(0,\cdot)\|_1$ and $mc = \int x g(0,x)\,dx$, in the sense that*

$$\lim_{t\to\infty}\int_{-\infty}^\infty h(x) g(t,x)\,dx = mh(c)$$

for all twice continuously differentiable functions $h : \mathbb{R} \to \mathbb{R}$.

Proof. Without loss of generality, $m = 1$ and $c = 0$ (since g may be scaled and translated). We first prove the statement on the support of $g(t,\cdot)$. In a similar way as above in the proof of Lemma 2.10, we see that

$$\frac{d}{dt}\int_b^\infty g(t,x)\,dx = \int_b^\infty \partial_x\big((W*g(t,\cdot))\cdot g(t,\cdot)\big)(x)\,dx = -(W*f(t,\cdot))(b)g(t,b).$$

So if $g(t,x) > 0$ for some $t > 0$ and $x > b$, we must have $g(\tau,b) > 0$ and $(W*g(\tau,\cdot))(b) < 0$ for some $0 < \tau < t$. Let t_0 be the infimum of $\tau > 0$ such that $g(\tau,b) > 0$. Then $g(t_0,x) = 0$ for $x \geq b$, and it follows that $(W*g(t_0,\cdot))(b) > 0$. By continuity, we will have $(W*g(\tau,\cdot))(b) > 0$ and $g(\tau,b) > 0$ for all sufficiently small $\tau > t_0$, so that the derivative above cannot be positive. So $g(t,x) > 0$ for some $t > 0$ and $x > b$ is not possible. This shows that $\operatorname{supp} g(t,\cdot) \subset\,]-\infty,b]$ for all $t > 0$. The argument for the lower bound is similar.

We now consider the second moment $M(t) = \int_{-\infty}^\infty x^2 g(t,x)\,dx$ w.r.t. the barycenter $c = 0$. There is $\mu > 0$ such that $xW(x) \geq \mu x^2$ for all $|x| \leq b-a$. Then

$$\begin{aligned}
\frac{d}{dt}M(t) &= \int_{-\infty}^\infty x^2\partial_x\big((W*g(t,\cdot))\cdot g(t,\cdot)\big)(x)\,dx\\
&= -2\int_{-\infty}^\infty x\,(W*g(t,\cdot))(x)g(t,x)\,dx\\
&= -2\int_{-\infty}^\infty\int_{-\infty}^\infty xW(x-y)g(t,y)g(t,x)\,dy\,dx\\
&= -\int_{-\infty}^\infty\int_{-\infty}^\infty (x-y)W(x-y)g(t,y)g(t,x)\,dy\,dx\\
&\leq -\mu\int_a^b\int_a^b (x-y)^2 g(t,y)g(t,x)\,dy\,dx\\
&= -2\mu\int_a^b x^2 g(t,x)\,dx = -2\mu M(t)\,.
\end{aligned}$$

This implies that $M(t) \leq e^{-2\mu}M(0)$; in particular, $M(t) \to 0$ as $t \to \infty$.

Now let $h \in \mathcal{C}^2(\mathbb{R})$. We can write $h(x) = h(0) + h'(0)x + h_2(x)$ where $h_2(0) = h_2'(0) = 0$. This implies that there is some $C > 0$ such that $|h_2(x)| \leq Cx^2$ for $x \in [a,b]$. We then have

$$\begin{aligned}
\Big|\int_{-\infty}^\infty h(x)g(t,x)\,dx - h(0)\Big| &= \Big|\int_{-\infty}^\infty (h(0) + h'(0)x + h_2(x))g(t,x)\,dx - h(0)\Big|\\
&= \Big|\int_a^b h_2(x)g(t,x)\,dx\Big| \leq \int_a^b |h_2(x)|g(t,x)\,dx \leq CM(t)\,,
\end{aligned}$$

and this tends to zero as $t \to \infty$. □

2.5. **Local masses and barycenters.**

For equation (1), the mass $\int_{S^1} f(t,\theta)\,d\theta$ is still an invariant, but there is no reasonable definition of a 'first moment'. (For this, one would need a function $F : S^1 \to \mathbb{R}$ that satisfies $F(\theta+a) = F(\theta)+a$ for all $\theta \in S^1$ and $a \in \mathbb{R}$. Such a function obviously does not exist.) However, we can define a localized version of a first moment.

Definition 2.12. Let $f : S^1 \to \mathbb{R}$ be continuous and nonnegative, and let $I \subset S^1$ be a closed interval. Let $I' = [a,b] \subset \mathbb{R}$ be an interval such that $p(I') = I$. We define the *local mass*

$$m(I,f) = \int_I f(\theta)\,d\theta = \int_a^b p^*(f)(x)\,dx \in \mathbb{R}$$

and, if $m(I,f) > 0$, the *local barycenter*

$$M(I,f) = p\Big(\frac{1}{m(I,f)} \int_a^b x p^*(f)(x)\,dx\Big) \in I\,.$$

The local barycenter does not depend on the choice of I': Any other choice has the form $I' + k$ with $k \in \mathbb{Z}$, and then we find that

$$\int_{a+k}^{b+k} x p^*(f)(x)\,dx = \int_a^b (x+k) p^*(f)(x)\,dx = \int_a^b x p^*(f)(x)\,dx + k m(I,f)\,,$$

so that the expression under $p(\cdot)$ changes by an integer, and the result is unchanged.

Lemma 2.13. *Let $f \geq 0$ be a solution of equation (1), and let $I \subset S^1$ be a closed interval. Then the local mass $m(I, f(t,\cdot))$ is time-invariant, provided there is no flow across the boundary of I: if $\alpha, \beta \in S^1$ are the endpoints of I, then we require*

$$(V * f(t,\cdot))(\alpha) f(t,\alpha) = (V * f(t,\cdot))(\beta) f(t,\beta) = 0$$

for all $t > 0$.

If in addition, there is no interaction with parts of f outside of I, meaning that

$$V(\theta - \psi) f(t,\theta) f(t,\psi) = 0 \qquad \text{if } \theta \in I \text{ and } \psi \notin I,$$

then the local barycenter $M(I, f(t,\cdot))$ is also time-invariant.

Proof. We have

$$\begin{aligned}\frac{d}{dt} m(I, f(t,\cdot)) &= \int_I \partial_\theta\big((V * f(t,\cdot)) f(t,\cdot)\big)(\theta)\,d\theta \\ &= (V * f(t,\cdot))(\beta) f(t,\beta) - (V * f(t,\cdot))(\alpha) f(t,\alpha) = 0\end{aligned}$$

and

$$\begin{aligned} m(I, f(t,\cdot)) & \frac{d}{dt} M(I, f(t,\cdot)) \\ &= \int_a^b x \partial_x\big((V * f(t,\cdot)) f(t,\cdot)\big)(x)\,dx \\ &= b(V * f(t,\cdot))(b) f(t,b) - a(V * f(t,\cdot))(a) f(t,a) - \int_a^b (V * f(t,\cdot))(x) f(t,x)\,dx \\ &= -\int_I \int_{S^1} V(\theta-\psi) f(t,\psi) f(t,\theta)\,d\psi\,d\theta \\ &= -\int_I \int_I V(\theta-\psi) f(t,\psi) f(t,\theta)\,d\psi\,d\theta \\ &= 0\,, \end{aligned}$$

because V is odd, compare Lemma 2.2. ☐

2.6. **Invariance of support.**

We consider equation (1) without diffusion on the circle (but what we say here is also valid for the equation on the real line).

Lemma 2.14. *Let* $A = I_1 \cup \ldots \cup I_n \subset S^1$ *be a disjoint union of closed intervals in* S^1*; write* $I_j = [\alpha_j, \beta_j]$*. Assume that for every continuous function* $h : S^1 \to \mathbb{R}_+$*, we have the implication*

$$h|_A = 0 \implies (V * h)(\alpha_j) > 0 \quad \text{and} \quad (V * h)(\beta_j) < 0 \quad \text{for all } 1 \leq j \leq n.$$

Let $f : [0, \infty[\times S^1 \to \mathbb{R}_+$ *be a solution of equation* (1) *with* $D = 0$ *such that* $f(0, \cdot)|_A = 0$*. Then* $f(t, \cdot)|_A = 0$ *for all* $t \geq 0$*.*

Equivalently, if $\operatorname{supp} f(0, \cdot) \subset \overline{S^1 \setminus A}$*, then* $\operatorname{supp} f(t, \cdot) \subset \overline{S^1 \setminus A}$ *for all* $t \geq 0$*.*

Proof. For the given solution f, let $\Phi : [0, \infty[\times S^1 \to S^1$ be the flow associated to $-(V * f)$,

$$\Phi(0, \theta) = \theta \quad \text{and} \quad \frac{\partial \Phi}{\partial t}(t, \theta) = -(V * f(t, \cdot))(\Phi(t, \theta)) \,.$$

Then it is readily checked that for all $\alpha, \beta \in S^1$, the integral

$$M(\alpha, \beta) = \int_{\Phi(t,\alpha)}^{\Phi(t,\beta)} f(t, \theta)\, d\theta$$

is independent of t. Write $\alpha_j(t) = \Phi(t, \alpha_j)$, $\beta_j(t) = \Phi(t, \beta_j)$ and

$$A(t) = \bigcup_{j=1}^{n} [\alpha_j(t), \beta_j(t)] \,,$$

then we have

$$\int_{A(t)} f(t, \theta)\, d\theta = 0$$

for all $t \geq 0$. Now assume that $f(t, \cdot)|_A$ is not identically zero for some $t > 0$. Then we must have that $A \not\subset A(t)$. Let t_0 be the infimum of all $t > 0$ such that $A \not\subset A(t)$. Then for some $1 \leq j \leq n$, we must have $\alpha_j(t_0) = \alpha_j$ and $\frac{d\alpha_j}{dt}(t_0) \geq 0$, or $\beta_j(t_0) = \beta_j$ and $\frac{d\beta_j}{dt}(t_0) \leq 0$. But in the first case

$$\frac{d\alpha_j}{dt}(t_0) = \frac{\partial \Phi}{\partial t}(t_0, \alpha_j) = -(V * f(t_0, \cdot))(\Phi(t_0, \alpha_j)) = -(V * f(t_0, \cdot))(\alpha_j) < 0 \,,$$

since $f(t_0, \cdot)|_A = 0$, and similarly in the second case $\frac{d\beta_j}{dt}(t_0) > 0$, leading to a contradiction. □

3. Stability of the constant solution

The constant function $f(\theta) = 1$ is a stationary solution of equation (1). The eigenvalues of the linearization around f are

$$c_k = 4\pi k(-\pi D k + v_k) \quad \text{for } k \in \mathbb{Z},$$

where $v_k = \int_0^{\frac{1}{2}} V(\theta) \sin(2\pi k \theta)\, d\theta$ (see (2) and (3)). Hence, the constant stationary solution is locally stable if $c_k < 0$ for all $k > 0$.

Remarks 3.1.
i) *Instability of the first mode*: If V is positive on $\left]0, \frac{1}{2}\right[$, i.e. all filaments attract each other, then the first mode is unstable for sufficiently small diffusion coefficient D: $c_1 > 0$ for $D \ll 1$. (This statement follows from $v_1 > 0$ and $c_1 = 4\pi(v_1 - \pi D)$.)
ii) *Instability of the second mode*: Assume that there exists $\theta_0 \in \left]0, \frac{1}{2}\right[$ such that $V(\theta) > 0$ on $]0, \theta_0[$, $V(\theta_0) = 0$ and $V(\theta) < 0$ on $\left]\theta_0, \frac{1}{2}\right[$. Moreover let (*) $V(\theta) \geq V(\frac{1}{2} - \theta)$ for $\theta \in \left]\min(\theta_0, \frac{1}{2} - \theta_0), \frac{1}{4}\right[$. Then the second mode is unstable for sufficiently small diffusion coefficient D, i.e. $c_2 > 0$ for $D \ll 1$. (This follows from Proposition 2.7 and i) because $\tilde{V}_2 > 0$ on $\left]0, \frac{1}{2}\right[$.)
Using exactly these assumptions on V, it has been shown by Primi et al. [16] that equation (1) has a $\frac{1}{2}$-periodic stationary solution with two equally large and very high maxima if the diffusion coefficient D is small enough.

Statement ii) can be interpreted in the following way: If there are already two peaks forming then attraction towards the nearer peak must be stronger than towards the second peak.
iii) If V is sufficiently regular (e.g. twice differentiable), then $k^2 v_k$ is bounded and c_k will be negative for $k \gg 0$. Therefore, higher modes tend to be linearly stable. Because periodicity is preserved and because there are no time-periodic solutions or chaos (see Proposition 2.4), instability of the k-th mode, i.e., $c_k > 0$, implies that there exist non-constant $\frac{1}{k}$-periodic stationary solutions.
iv) Chayes and Panferov [3] proved (on tori) that under the regularity assumption $\sum_k |v_k| < \infty$ local linear stability implies that there is a non-trivial basin of attraction for 1.

Moreover, Chayes and Panferov show that for small enough interaction (i.e. small parameter in [3]) the constant is the only minimizer of the 'free energy' functional. We prove now the *dynamical* fact that all solutions converge to the constant 1 if the interaction V (represented by v_k) is small compared to D.

Theorem 3.2 (Global stability of the constant solution)**.** *Let $D \geq 0$. Assume that*

$$\rho^2 = \sum_{k>0} \frac{k(k^2-1)}{6} v_k^2 < \infty$$

(this is the case when V is twice continuously differentiable, for example). If

$$v_k < \pi D\, k - \frac{\rho}{k} \qquad \text{for all } k \geq 1,$$

then every nonnegative initial function $f(0,\cdot) \in \mathcal{C}(S^1)$ of mass 1 converges to the constant function 1 — there is some $c > 0$ such that

$$\left\|\partial_\theta^n (f(t,\cdot) - 1)\right\|_\infty = O_n\left(e^{-ct}\right)$$

for all $n \geq 0$.

Proof. Recall that we assume $f_0 = 1$. We scale time by a factor 4π and set $\delta = \pi D$ in the Fourier transformed system (2) to get

$$\dot{f}_k = k\Big((-\delta k + v_k) f_k + \sum_{l \in \mathbb{Z}\setminus\{0,k\}} v_l f_l f_{k-l}\Big). \tag{7}$$

Because $f_{-k} = \bar{f}_k$ (f is real) and $v_{-k} = -v_k \in \mathbb{R}$, we see that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\sum_{k\geq 1}\frac{1}{k}|f_k|^2 &= \sum_{k\geq 1}\frac{1}{k}\operatorname{Re}(\bar{f}_k \dot{f}_k) \\
&= \sum_{k\geq 1}(v_k - k\delta)|f_k|^2 + \operatorname{Re}\Big(\sum_{k\geq 1}\sum_{l>k} v_l f_l \bar{f}_k \bar{f}_{l-k}\Big) \\
&\qquad + \operatorname{Re}\Big(\sum_{k\geq 1}\sum_{1\leq l<k} v_l f_l \bar{f}_k f_{k-l} - \sum_{k\geq 1}\sum_{l\geq 1} v_l \bar{f}_l \bar{f}_k f_{k+l}\Big) \\
&= \sum_{k\geq 1}(v_k - k\delta)|f_k|^2 + \operatorname{Re}\Big(\sum_{k,m\geq 1} v_{k+m} f_{k+m} \bar{f}_k \bar{f}_m\Big)
\end{aligned}$$

To justify the last equality, note that, setting $k \leftarrow k+l$, we have

$$\operatorname{Re}\Big(\sum_{k\geq 1}\sum_{1\leq l<k} v_l f_l \bar{f}_k f_{k-l}\Big) = \operatorname{Re}\Big(\sum_{k,l\geq 1} v_l f_l f_k \bar{f}_{k+l}\Big) = \operatorname{Re}\Big(\sum_{k,l\geq 1} v_l \bar{f}_l \bar{f}_k f_{k+l}\Big),$$

In the remaining sum, we have set $l \leftarrow k+m$. We can estimate it as follows.

$$
\begin{aligned}
&\Big|\mathrm{Re}\Big(\sum_{k,m\geq 1} v_{k+m} f_{k+m} \bar{f}_k \bar{f}_m\Big)\Big|^2 \\
&\quad \leq \Big|\sum_{k,m\geq 1} v_{k+m} f_{k+m} \bar{f}_k \bar{f}_m\Big|^2 \\
&\quad = \Big|\sum_{k,m\geq 1} \sqrt{km(k+m)}\, v_{k+m} \frac{f_{k+m}}{\sqrt{k+m}} \frac{\bar{f}_k}{\sqrt{k}} \frac{\bar{f}_m}{\sqrt{m}}\Big|^2 \\
&\quad \leq \Big(\sum_{k\geq 1} \frac{1}{k}|f_k|^2\Big)\Big(\sum_{k\geq 1}\Big|\sum_{m\geq 1} \sqrt{km(k+m)}\, v_{k+m} \frac{f_{k+m}}{\sqrt{k+m}} \frac{\bar{f}_m}{\sqrt{m}}\Big|^2\Big) \\
&\quad \leq \Big(\sum_{k\geq 1} \frac{1}{k}|f_k|^2\Big)\Big(\sum_{k\geq 1}\Big(\sum_{m\geq 1} \frac{1}{m}|f_m|^2\Big)\Big(\sum_{m\geq 1} km(k+m) v_{k+m}^2 \frac{1}{k+m}|f_{k+m}|^2\Big)\Big) \\
&\quad = \Big(\sum_{k\geq 1} \frac{1}{k}|f_k|^2\Big)^2\Big(\sum_{l\geq 1}\Big(\sum_{k=0}^{l} k(l-k)\Big) l v_l^2 \frac{1}{l}|f_l|^2\Big) \\
&\quad \leq \Big(\sum_{k\geq 1} \frac{1}{k}|f_k|^2\Big)^2\Big(\sum_{l\geq 1} \frac{l^2(l^2-1)}{6} v_l^2 \frac{1}{l}|f_l|^2\Big) \qquad (8)
\end{aligned}
$$

Inequality (8) implies

$$
\frac{1}{2}\frac{d}{dt}\|f\|^2 \leq \Big(\sqrt{\sum_{l\geq 1} \frac{l^2(l^2-1)}{6} v_l^2 \frac{1}{l}|f_l|^2} + \max_{k\geq 1} k(v_k - k\delta)\Big)\|f\|^2 \leq \big(\rho + \max_{k\geq 1} k(v_k - k\delta)\big)\|f\|^2. \qquad (9)
$$

(with $\|f\|^2 = \sum_{k\geq 1} \frac{1}{k}|f_k|^2$ as before and using $|f_l| \leq 1$).

Set

$$
c = -\big(\max_{k\geq 1} k(v_k - k\delta) + \rho\big) > 0.
$$

It follows that

$$
\|f(t)\|^2 \leq \|f(0,\cdot)\|^2 e^{-2ct}
$$

for $t \geq 0$. Since $\frac{1}{k}|f_k|^2 \leq \|f\|^2$, this implies that

$$
|f_k(t)| \leq \sqrt{k}\|f(0,\cdot)\| e^{-ct} \quad \text{for } k \geq 1 \text{ and } t \geq 0.
$$

We need a lemma.

Lemma 3.3. *Assume that $|f_k(t)| \leq Ck^\alpha e^{-ct}$ for all $k \geq 1$ and all $t \geq 0$, where $C > 0$ and $\alpha \leq \frac{1}{2}$ are constants. Then for any $t_0 > 0$, there is a constant $C' > 0$ (depending on t_0) such that $|f_k(t)| \leq C' k^{\alpha-2} e^{-ct}$ for all $k \geq 1$ and all $t \geq t_0$.*

Proof of Lemma 3.3. The quadratic part of the right hand side of the differential equation (7) for f_k is

$$
R_k = \sum_{0<l<k} v_l f_l f_{k-l} + \sum_{l>k} v_l f_l \bar{f}_{l-k} - \sum_{l>0} v_l \bar{f}_l f_{k+l}
$$

We estimate R_k:

$$
\begin{aligned}
|R_k(t)| &\leq C^2 e^{-2ct}\Big(\sum_{l>0, l\neq k} |v_l|\, l^\alpha\, |l-k|^\alpha + \sum_{l>0} |v_l|\, l^\alpha\, |l+k|^\alpha\Big) \\
&\leq C^2 k^\alpha e^{-2ct} \sum_{l>0} C_1 l |v_l| \leq C_2 k^\alpha e^{-2ct} \leq C_2 k^\alpha e^{-ct}.
\end{aligned}
$$

Here we use that $\sum_{l>0} l|v_l| < \infty$ and that $l^\alpha |l \pm k|^\alpha \le C_1 k^\alpha l$ for some constant C_1 only depending on α. Write $c'_k = k(v_k - k\delta) = c_k/(4\pi) < 0$. Then we have

$$\dot{f}_k - c'_k f_k = R_k$$

and therefore

$$f_k(t) = e^{c'_k t} f_k(0) + \int_0^t e^{c'_k(t-\tau)} R_k(\tau)\, d\tau.$$

The integral is bounded by

$$C_2 k^\alpha \left| \frac{e^{-ct} - e^{c'_k t}}{c + c'_k} \right| \le C_3 k^{\alpha-2} e^{-ct}$$

for some constant C_3 (note that $-c'_k \gg k^2$ and that $|c + c'_k| = |c'_k| - c > \rho \|f(0,\cdot)\|^2 > 0$). This gives

$$|f_k(t)| \le \Big(k^{2-\alpha} e^{(c'_k + c)t} |f_k(0)| + C_3 \Big)\, k^{\alpha-2} e^{-ct}.$$

Since $c'_k + c \le -\text{const.}\, k^2$, the first summand in brackets is bounded uniformly in $k > 0$ for $t \ge t_0 > 0$. This finishes the proof of the lemma. □

Repeated application of the lemma then shows that, given $N > 0$ and $t_0 > 0$, there is a constant $C_N > 0$ such that

$$|f_k(t)| \le C_N k^{-N} e^{-ct} \quad \text{for all } k \ge 1 \text{ and all } t \ge t_0.$$

This implies

$$\|f(\cdot,t) - 1\|_\infty \le 2 \sum_{k \ge 1} |f_k(t)| = O\big(e^{-ct}\big).$$

Similarly, for any $n \ge 1$, we obtain

$$\|\partial_\theta^n f(\cdot,t)\|_\infty \le 2(2\pi)^n \sum_{k \ge 1} k^n |f_k(t)| = O\big(e^{-ct}\big).$$

□

4. Convergence to peak solutions in case of small initial support and no diffusion

In this section we assume that there is *no random turning*, i.e., $D = 0$ in (1). Then sums of delta peaks can be stationary solutions. To make this precise, we have to define the right hand side of equation (1) for suitable distributions on the circle. Compare also Carrillo et.al. [1] for definition and existence of weak solutions on $\mathbb{R}^n$.

The kind of distribution we are mostly interested in are (positive) *measures*, but it turns out that it is advantageous to use *differentiable measures* instead. The main reason for this is that the map $S^1 \to \mathcal{D}^0(S^1)$, $\psi \mapsto \delta_\psi$, is continuous, but not differentiable (since the derivative at zero would have to be $-\delta'_0$). As a map to $\mathcal{D}^1(S^1)$, it becomes differentiable, though.

Let $k \ge 0$. The space $\mathcal{D}^k(S^1)$ is the dual space of the space $\mathcal{C}^k(S^1)$ of k times continuously differentiable functions on the circle S^1. The elements of $\mathcal{D}^0(S^1)$ are called *measures*, and the elements of $\mathcal{D}^1(S^1)$ are called *differentiable measures* on S^1. By standard theory (see for example [11]), $\mathcal{D}^k(S^1)$ can be identified with the subspace of $\mathcal{D}(S^1)$ (which is the dual of $\mathcal{C}^\infty(S^1)$) consisting of distributions of order $\le k$, i.e. $f \in \mathcal{D}^k(S^1)$ if and only if

$$\text{there is } C > 0 \text{ such that} \qquad \big|\langle f, h\rangle\big| \le C \sum_{j=0}^{k} \|h^{(j)}\|_\infty \quad \text{for all } h \in \mathcal{C}^\infty(S^1).$$

In particular, we can consider $\mathcal{D}^0(S^1)$ as a subspace of $\mathcal{D}^1(S^1)$. $\mathcal{D}(S^1)$ can be identified with the space of 1-periodic distributions on $\mathbb{R}$, compare [11].

A distribution $f \in \mathcal{D}^k(S^1)$ is *non-negative* if $\langle f, h\rangle \geq 0$ for every non-negative function $h \in \mathcal{C}^k(S^1)$. We write $\mathcal{D}^k_+(S^1)$ for the set of non-negative distributions in $\mathcal{D}^k(S^1)$. Note that for $f \in \mathcal{D}^k_+(S^1)$ and test functions $h_1 \leq h_2$ we have $\langle f, h_1\rangle \leq \langle f, h_2\rangle$.

The *support* $\operatorname{supp} f$ of $f \in \mathcal{D}^k(S^1)$ is the smallest closed subset of S^1 outside of which $f = 0$.

Let $\psi \in S^1$. The *delta distribution* $\delta_\psi \in \mathcal{D}^0_+(S^1)$ is defined by $\langle \delta_\psi, h\rangle = h(\psi)$ where h is a test function.

The *mass* of a distribution $f \in \mathcal{D}(S^1)$ is defined as $\int_{S^1} f(\theta)\, d\theta = \langle f, 1\rangle$. We still assume that the mass of solutions is 1 (sometimes we mention it again to clarify statements).

The *convolution* of a distribution $f \in \mathcal{D}^k(S^1)$ with a function $V \in \mathcal{C}^l(S^1)$ with $l \geq k$ is defined as

$$(V * f)(\theta) = \langle f, V(\cdot - \psi)\rangle$$

for $\theta \in S^1$. It is known that $V * f \in \mathcal{C}^{l-k}(S^1)$ (see [11]). For example, $(V * \delta_\psi)(\theta) = V(\theta - \psi)$ for $\theta, \psi \in S^1$.

Convergence $f_n \xrightarrow{\mathcal{D}} f$ in $\mathcal{D}^k(S^1)$ as $n \to \infty$ means that $\langle f_n, h\rangle \to \langle f, h\rangle$ as $n \to \infty$ for all test functions $h \in \mathcal{C}^k(S^1)$.

Now, based on these considerations we define the transport term of (1) as

$$\Big\langle \frac{\partial}{\partial\theta}\left((V * f)\, f\right), h\Big\rangle = -\langle f, (V * f)h'\rangle. \tag{10}$$

Proposition 4.1. *Let $V \in \mathcal{C}^2(S^1)$ be odd and consider equation (1) with $D = 0$.*
i) A single peak $f = \delta_\psi \in \mathcal{D}^1_+(S^1)$ with $\psi \in S^1$ is a stationary solution of (1) with mass 1.
ii) Let $V(\theta_0) = 0$ for fixed $0 < \theta_0 \leq \frac{1}{2}$. Two peaks with arbitrary *masses and distance θ_0 are a stationary solution, i.e. $f = m_1\delta_\psi + m_2\delta_{\psi+\theta_0} \in \mathcal{D}^1_+(S^1)$ is a stationary solution of (1) for any $\psi \in S^1$ and $m_1, m_2 > 0$. If $m_1 + m_2 = 1$ then f has mass 1.*
iii) $n \geq 3$ peaks with equal *masses and equal distances are a stationary solution: For $1 \leq j \leq n$ let $\psi_j \in S^1$ with $\psi_{j+1} - \psi_j = \frac{1}{n}$ (where $\psi_{n+1} = \psi_1$). Then $f = \frac{1}{n}\sum_{j=1}^n \delta_{\psi_j} \in \mathcal{D}^1_+(S^1)$ is a stationary solution of (1) with mass 1.*

Note that $n \geq 3$ peaks with *different* masses are in general no stationary solution contrary to the "degenerate" case $n = 2$ (where $V(\frac{1}{2}) = 0$ holds always). However, the situation changes if V has suitably spaced zeros, e.g. if $V(\frac{1}{4}) = 0$. Then stationary solutions consisting of four peaks with distance $\frac{1}{4}$ and possibly *different* masses occur.

Proof. Let $h \in \mathcal{C}^1(S^1)$ be a test function.

i) Using (10) and $V(0) = 0$, we get $\langle \delta_\psi, (V * \delta_\psi)h'\rangle = V(0)h'(\psi) = 0$.

ii) Let $\psi_2 = \psi + \theta_0$. Using (10) and $V(0) = V(\theta_0) = V(-\theta_0) = 0$, we get

$$\begin{aligned}
&\langle m_1\delta_\psi + m_2\delta_{\psi_2}, (V * (m_1\delta_\psi + m_2\delta_{\psi_2}))h'\rangle \\
&\quad = m_1\langle \delta_\psi, (V * (m_1\delta_\psi + m_2\delta_{\psi_2}))h'\rangle + m_2\langle \delta_{\psi_2}, (V * (m_1\delta_\psi + m_2\delta_{\psi_2}))h'\rangle \\
&\quad = m_1(m_1V(0) + m_2V(-\theta_0))h'(\psi) + m_2(m_1V(\theta_0) + m_2V(0))h'(\psi_2) = 0.
\end{aligned}$$

iii) This follows immediately from statement i) and Proposition 2.7. ∎

In the following theorems and corollaries we show that solutions converge to sums of peaks if the support of the initial function is either sufficiently small (in the case of a single peak) or such that particles in different intervals do not interact. Fellner and Raoul [5] show similar results for the transport equation on $\mathbb{R}$ by linearization. Since they transform the PDE to an integro-differential equation by using the 'pseudo-inverse' of the solution we think that their methods do not apply to the equation on S^1.

Theorem 4.2 (Small initial support and single peak)**.** *Let $V \in \mathcal{C}^2(S^1)$ be odd with $V'(0) > 0$ and $V > 0$ on $]0, \theta_v[$, where $0 < \theta_v \leq \frac{1}{2}$. Assume that $f \geq 0$ is a solution of (1) with $D = 0$ such that $\operatorname{supp} f(0, \cdot) \subset I$ where $I \subset S^1$ is a closed interval with $\operatorname{Vol}(I) < \theta_v$ and such that $m(I, f(0, \cdot)) = 1$.*

Then $\operatorname{supp} f(t, \cdot) \subset I$ for all $t \geq 0$, and $f(t, \cdot)$ converges to the delta distribution δ_M, where $M = M(I, f(0, \cdot))$ is the local barycenter of $f(0, \cdot)$ on I.

Proof. We lift I to an interval $I' = [a, b] \subset \mathbb{R}$ and let $\ell = b - a < \theta_v$. Then $f(0, \cdot) = p_*(g_0)$ for a function $g_0 : \mathbb{R} \to \mathbb{R}$ with $\operatorname{supp} g_0 \subset I'$.

We consider equation (6), where we take $W = p^*(V)$ on $[-\ell, \ell]$ and extend it to all of $\mathbb{R}$ in such a way that it is odd and satisfies $W(x) > 0$ for $x > 0$ (which is possible since $p^*(V) > 0$ on $]0, \ell]$). Let g be the solution of equation (6) with $D = 0$ such that $g(0, \cdot) = g_0$. By Proposition 2.11, $\operatorname{supp} g(t, \cdot) \subset I'$ for all $t \geq 0$. So the function $(W * g(t, \cdot))g(t, \cdot)$ appearing on the right hand side of equation (6) will always be equal to $(p^*(V) * g(t, \cdot))g(t, \cdot)$ (since $W(x - y) = p^*(V)(x - y)$ when $x, y \in I'$). This means that g will also be the solution of equation (6), if we use $p^*(V)$ instead of W. By Proposition 2.8, we then have $f(t, \cdot) = p_*(g(t, \cdot))$ for all $t \geq 0$. In particular, $\operatorname{supp} f(t, \cdot) \subset p(\operatorname{supp} g(t, \cdot)) \subset p(I') = I$. By Proposition 2.11, we also know that $g(t, \cdot)$ converges to $\delta_{M'}$, where $M' = \int_{\mathbb{R}} x g(0, x)\, dx$, so $f(t, \cdot) = p_*(g(t, \cdot))$ will converge to δ_M, since $M = p(M')$. □

Example 4.3. Initial growth of an already sharp peak may be also seen if $0 < D \ll 1$ even if no single peak solution is expected, see Figure 1. In Figure 1 the interaction function is $\frac{1}{2}$-periodic, namely $V(\theta) = \sin(4\pi\theta)$. By Proposition 2.6 any stationary solution must be $\frac{1}{2}$-periodic (and such stationary solutions exist and are expected to be stable). Therefore, with positive diffusion $D > 0$ a single peak solution is not expected for (1), but as shown in the figure a sharp peak grows initially. Indeed, we did not see the development of a second peak although we had the program run up to times larger than 140.

Therefore, the behavior observed here must be an artifact of the numerics. We think that the explanation is that the time scale for the transition from one peak to two peaks should be roughly of the order of $e^{1/D}$, so the rate of change would be of order $e^{-1/D}$, which is numerically zero if D is as small as in the example.

The next corollary follows directly from Theorem 4.2 and Proposition 2.7. The assumptions on V_n imply also that the n-th eigenvalue is positive (Corollary 3.1).

Corollary 4.4. *Let $n \geq 1$, $0 < \theta_v < \frac{1}{2n}$, and let $V \in \mathcal{C}^2(S^1)$ be odd and such that $V_n'(0) > 0$ and $V_n > 0$ on $]0, \theta_v[$. Let $f \geq 0$ be a solution of (1) with $D = 0$ such that $f(0, \cdot)$ is $\frac{1}{n}$-periodic, and assume that there exists an interval $I \subset S^1$ with $\operatorname{Vol}(I) < \theta_v$ such that $\operatorname{supp} f(0, \cdot) \subset \bigcup_{j=0}^{n-1}(I + \frac{j}{n})$.*

Then the solution f converges to n peaks of equal masses at equal distances:

$$f(t, \cdot) \xrightarrow{\mathcal{D}} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{M + \frac{j}{n}} \qquad \text{as } t \to \infty, \qquad \text{where } M = M(I, f(0, \cdot)).$$

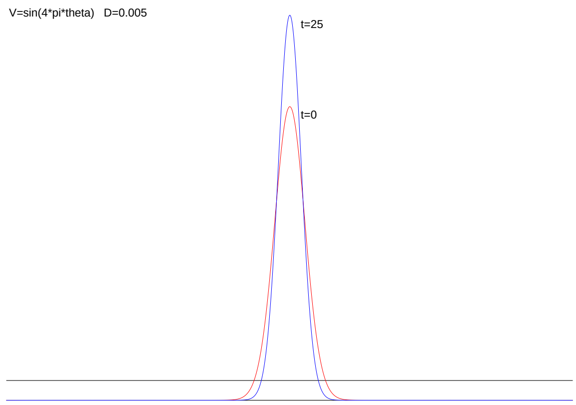

FIGURE 1. $V(\theta) = \sin(4\pi\theta)$, $D = 0.005$, and $f(0,\cdot)$ is the stationary solution of (1) for $V(\theta) = \sin(2\pi\theta)$, $D = 0.005$. The numerical algorithm is described in Section 6 (Fourier based method with 61 Fourier coefficients). (The horizontal lines are 0 and 1.)

The following corollary states that several peaks at random distances may form if V is zero in a neighborhood of $\frac{1}{2}$. There is, however, a minimal distance between them. This result may be of interest in relation to results of Chayes and Panferov [3] since these authors assume interaction potentials with non-trivial compact support.

Corollary 4.5. *Let $V \in \mathcal{C}^2(S^1)$ be odd and $0 < \theta_v < \frac{1}{2}$ such that $V > 0$ on $]0, \theta_v[$ and $V(\theta) = 0$ for $\theta_v \le |\theta| \le \frac{1}{2}$. Let $n \ge 1$ and $f(0,\cdot) \in \mathcal{C}^+(S^1)$, $\operatorname{supp} f(0,\cdot) \subset \bigcup_{j=1}^n I_j$ where $\operatorname{Vol}(I_j) < \theta_v$ and $\operatorname{dist}(I_j, I_k) > \theta_v$ for all $1 \le j \ne k \le n$. Define $m_j(t) = m(I_j, f(t,\cdot))$ and $M_j(t) = M(I_j, f(t,\cdot))$.*

Then $\operatorname{supp}(f(t,\cdot)) \subset \bigcup_j I_j$ for all $t \ge 0$ and the masses m_j as well as the barycenters M_j are constant in t. The solution f converges to a sum of delta peaks:

$$f(t,\cdot) \xrightarrow{\mathcal{D}} \sum_{j=1}^{n} m_j\, \delta_{M_j} \qquad \text{as } t \to \infty.$$

Proof. As long as the support of $f(t,\cdot)$ stays contained in the union of the I_j, the evolution of f on each of the I_j proceeds independently, since the part of f contained in the other intervals does not contribute to the right hand side of equation (1). But then Theorem 4.2 shows that the part that starts in I_j stays in I_j and converges to a delta peak as stated. □

In the following theorem we are interested in convergence to two peaks, but Proposition 2.7 cannot be used since $f(0,\cdot)$ is not necessarily $\frac{1}{2}$-periodic. Neither is V $\frac{1}{2}$-periodic in general.

We first prove a lemma that allows us to show convergence to the delta-distribution if mass is constant and second moments converge to zero.

Lemma 4.6. *Let $I \subset S^1$ be a closed interval. Let $f_n \in \mathcal{D}_+^1(S^1)$ with $\operatorname{supp} f_n \subset I$ and $\langle f_n, 1\rangle = 1$ for all $n \ge 1$. Let $M \in I$, and let $q_M : S^1 \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function satisfying $q_M(\theta) = (\theta - M)^2$ for all $\theta \in I$. If $\langle f_n, q_M\rangle \to 0$ as $n \to \infty$, then $f_n \xrightarrow{\mathcal{D}} \delta_M$ as $n \to \infty$.*

Note that the same conclusion is valid when we only assume that $q_M(\theta) \ge c(\theta - M)^2$ for all $\theta \in I$ with some $c > 0$.

Proof. Let $\ell_M : S^1 \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function such that $\ell_M(\theta) = \theta - M$ for $\theta \in I$, and define $a_n = \langle f_n, \ell_M\rangle$ and $b_n = \langle f_n, q_M\rangle$ Then by the Cauchy-Schwarz inequality (applied to the inner product $(g,h) \mapsto \langle f_n, gh\rangle$ for functions $g, h : I \to \mathbb{R}$, concretely with $g = 1$ and $h = \ell_M$) we have $a_n^2 \le b_n$. Since $b_n \to 0$, we must have $a_n \to 0$ as well. Let $h \in \mathcal{C}^1(S^1)$ be a test function. We can write

$$h(\theta) = h(M) + h'(M)\ell_M(\theta) + r(\theta)q_M(\theta)$$

for $\theta \in I$, with $|r(\theta)| \le C = \frac{1}{2}\max_I |h''|$. We then have

$$\begin{aligned} \big|\langle f_n - \delta_M, h\rangle\big| &= \big|\langle f_n, h(M) + h'(M)\ell_M + rq_M\rangle - h(M)\big| \\ &= \big|h'(M)\langle f_n(x), \ell_M\rangle + \langle f_n(x), rq_M\rangle\big| \\ &\le |h'(M)|a_n + Cb_n \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

□

The final positions $\bar{M}_0$ and $\bar{M}_1$ of the two peaks in the theorem below are obtained from the special case when $V(p(x)) = cx$ with $c > 0$ in an interval around zero and V is $\frac{1}{2}$-periodic. In this case one gets equations $\frac{dM_j}{dt}(t) = -cM_j(t)$ (with $M_j(t)$ defined as in the proof below), so that $M_j(t) \to 0$, justifying the choice of $\bar{M}_j$.

Theorem 4.7 (Small initial supports and two peaks)**.** *Let $V \in \mathcal{C}^2(S^1)$ be odd with $V'(0) > 0$ and $V'(\frac{1}{2}) > 0$, and assume that there exist $0 < \theta_1 \le \theta_2 < \frac{1}{2}$ such that $V > 0$ on $]0, \theta_1[$ and $V < 0$ on $\big]\theta_2, \frac{1}{2}\big[$.*

Let $f \ge 0$ be a solution of equation (1) with $D = 0$ such that $f(0,\cdot) \in \mathcal{C}^+(S^1)$ with $\int_{S^1} f(0,\theta)\,d\theta = 1$ and $\operatorname{supp} f(0,\cdot) \subset I_0 \cup I_1$ where $I_0 \subset S^1$ is a closed interval such that $\operatorname{Vol}(I_0) < \min\{\theta_1, \frac{1}{2} - \theta_2\}$ and $I_1 = I_0 + \frac{1}{2}$. Then $\operatorname{supp} f(t,\cdot) \subset I_0 \cup I_1$ for all $t \ge 0$.

Let I be a closed interval in S^1 containing $I_0 \cup I_1$. Define the local masses $m_j(t) = m(I_j, f(t,\cdot))$, and let $M(t) = M(I, f(t,\cdot))$ be the local barycenter on I. Then $m_0(t)$, $m_1(t)$ and $M(t)$ are constant in time; we write m_0, m_1 and M for their values. Define

$$\bar{M}_0 = M + \tfrac{1}{2}m_1 \qquad \text{and} \qquad \bar{M}_1 = M - \tfrac{1}{2}m_0 = \bar{M}_0 - \tfrac{1}{2}.$$

Then $f(t,\cdot)$ converges to a sum of two opposite peaks:

$$f(t,\cdot) \to m_0\delta_{\bar{M}_0} + m_1\delta_{\bar{M}_1} \qquad \text{as} \qquad t \to \infty.$$

Proof. We first show that $\operatorname{supp} f(t,\cdot) \subset I_0 \cup I_1$ for all $t \ge 0$. Let $I_0 = [\alpha, \beta]$, then $I_1 = [\alpha', \beta'] = [\alpha + \frac{1}{2}, \beta + \frac{1}{2}]$; let $\varepsilon = \operatorname{Vol}(I_0) = \beta - \alpha < \min\{\theta_1, \frac{1}{2} - \theta_2\}$. Let $h : S^1 \to \mathbb{R}_+$ with $\operatorname{supp} h \subset I_0 \cup I_1$. Because $V > 0$ on $]0, \theta_1[\cup \big]\frac{1}{2}, 1 - \theta_2\big[$, we see that

$$(V * h)(\beta) = \int_\alpha^\beta V(\underbrace{\beta - \psi}_{0<\bullet<\theta_1})h(\psi)\,d\psi + \int_{\alpha'}^{\beta'} V(\underbrace{\beta - \psi}_{\frac{1}{2}<\bullet<1-\theta_2})h(\psi)\,d\psi > 0.$$

Similarly,

$$(V * h)(\alpha) = \int_\alpha^\beta V(\underbrace{\alpha - \psi}_{-\theta_1<\bullet<0})h(\psi)\,d\psi + \int_{\alpha'}^{\beta'} V(\underbrace{\alpha - \psi}_{\theta_2<\bullet<\frac{1}{2}})h(\psi)\,d\psi < 0,$$

because V is negative on both intervals. In the same way, we get $(V*h)(\alpha') > 0$ and $(V*h)(\beta') < 0$. By Lemma 2.14 it follows that $\operatorname{supp} f(t,\cdot) \subset I_0 \cup I_1$ for all $t \ge 0$.

By Lemma 2.13, the local masses $m_j(t)$ are then constant, and the same is true for $M(t)$ (since $f(t,\cdot)=0$ on $S^1\setminus I$ for all $t\geq 0$). We now define local first and second moments by

$$M_j(t)=\int_{I_j}(\theta-\bar{M}_j)f(t,\theta)\,d\theta \qquad \text{and} \qquad m_{2,j}(t)=\int_{I_j}(\theta-\bar{M}_j)^2 f(t,\theta)\,d\theta \quad \text{for } j\in\{0,1\}.$$

Note that the expression $\theta-\bar{M}_j$ makes sense on I_j (even on I: we lift to a suitable interval in $\mathbb{R}$ and compute the difference there). The definitions imply that $M_0(t)+M_1(t)=M-m_0\bar{M}_0-m_1\bar{M}_1=0$ for all $t\geq 0$. Let $m_2(t)=m_{2,0}(t)+m_{2,1}(t)$. We will show that $m_2(t)\to 0$ as $t\to\infty$. The time derivative of m_2 is (after integration by parts)

$$\begin{aligned}\frac{dm_2}{dt}(t)=&-2\Big(\int_{I_0}\int_{I_0}+\int_{I_0}\int_{I_1}\Big)V(\theta-\psi)(\theta-\bar{M}_0)f(t,\psi)f(t,\theta)\,d\psi\,d\theta\\&-2\Big(\int_{I_1}\int_{I_0}+\int_{I_1}\int_{I_1}\Big)V(\theta-\psi)(\theta-\bar{M}_1)f(t,\psi)f(t,\theta)\,d\psi\,d\theta.\end{aligned}$$

To estimate this, we observe that there is $b>0$ such that

$$V(\phi)\phi\geq b\phi^2 \qquad \text{and} \qquad V(\phi+\tfrac{1}{2})\phi\geq b\phi^2 \qquad \text{for all } \phi\in[-\varepsilon,\varepsilon].$$

This is because $V>0$ on $]0,\varepsilon]$ and on $\big]\frac{1}{2},\frac{1}{2}+\varepsilon\big]$ and because $V(0)=V(\frac{1}{2})=0$, $V'(0)>0$ and $V'(\frac{1}{2})>0$. We now bound the various integrals from below. For the first, we find

$$\begin{aligned}&2\int_{I_0}\int_{I_0}V(\theta-\psi)(\theta-\bar{M}_0)f(t,\psi)f(t,\theta)\,d\psi\,d\theta\\&=\int_{I_0}\int_{I_0}V(\theta-\psi)(\theta-\bar{M}_0)f(t,\psi)f(t,\theta)\,d\psi\,d\theta+\int_{I_0}\int_{I_0}V(\psi-\theta)(\psi-\bar{M}_0)f(t,\psi)f(t,\theta)\,d\psi\,d\theta\\&=\int_{I_0}\int_{I_0}V(\theta-\psi)(\theta-\psi)f(t,\psi)f(t,\theta)\,d\psi\,d\theta\\&\geq b\int_{I_0}\int_{I_0}(\theta-\psi)^2f(t,\psi)f(t,\theta)\,d\psi\,d\theta\\&=b\int_{I_0}\int_{I_0}\big((\theta-\bar{M}_0)-(\psi-\bar{M}_0)\big)^2f(t,\psi)f(t,\theta)\,d\psi\,d\theta\\&=2b\big(m_{2,0}(t)m_0-M_0(t)^2\big).\end{aligned}$$

In the same way, we find for the fourth integral that

$$\int_{I_1}\int_{I_1}V(\theta-\psi)(\theta-\bar{M}_1)f(t,\psi)f(t,\theta)\,d\psi\,d\theta\geq b\big(m_{2,1}(t)m_1-M_1(t)^2\big).$$

The remaining two integrals are estimated together, as follows.

$$\begin{aligned}&\int_{I_0}\int_{I_1}V(\theta-\psi)(\theta-\bar{M}_0)f(t,\psi)f(t,\theta)\,d\psi\,d\theta+\int_{I_1}\int_{I_0}V(\theta-\psi)(\theta-\bar{M}_1)f(t,\psi)f(t,\theta)\,d\psi\,d\theta\\&\quad=\int_{I_0}\int_{I_1}V(\theta-\psi)\big((\theta-\bar{M}_0)-(\psi-\bar{M}_1)\big)f(t,\psi)f(t,\theta)\,d\psi\,d\theta\\&\quad\geq b\int_{I_0}\int_{I_1}\big((\theta-\bar{M}_0)-(\psi-\bar{M}_1)\big)^2f(t,\psi)f(t,\theta)\,d\psi\,d\theta\\&\quad=b\big(m_{2,0}(t)m_1-2M_0(t)M_1(t)+m_{2,1}(t)m_0\big).\end{aligned}$$

Adding up, we find that (recalling that $M_0(t)+M_1(t)=0$)

$$\frac{dm_2}{dt}(t)\leq -2b\big((m_{2,0}(t)+m_{2,1}(t))(m_0+m_1)-M_0(t)^2-2M_0(t)M_1(t)-M_1(t)^2\big)=-2bm_2(t).$$

This shows that $m_2(t) \le e^{-2bt} m_2(0)$, and since $m_2(t) \ge 0$, this implies $m_2(t) \to 0$ as $t \to \infty$. So the local second moments $m_{2,0}(t)$ and $m_{2,1}(t)$ tend to zero as well. Using Lemma 4.6 on the intervals I_0, I_1 separately, it follows that for $t \to \infty$ the solution converges to two peaks,

$$f(t,\cdot) \xrightarrow{\mathcal{D}} m_0 \delta_{\bar{M}_0} + m_1 \delta_{\bar{M}_1},$$

where the distance between the peaks is $\bar{M}_0 - \bar{M}_1 = \frac{1}{2}$. □

5. Linear stability of peaks

5.1. **Instability conditions.**

We start this section by following the 'peak ansatz' of Mogilner et.al. [15]. The initial distribution is a sum of $n \ge 2$ peaks at positions $\theta_j(0) \in S^1$ and with masses $m_j > 0$ where $\sum_{j=1}^n m_j = 1$ (different masses are a generalization of [15]). The solution keeps this form, $f(t,\theta) = \sum_{j=1}^n m_j \delta_{\theta_j(t)}(\theta)$, and the positions $\theta_j(t)$ satisfy the following system of ordinary differential equations.

$$\frac{d\theta_j}{dt}(t) = -\sum_{k=0}^{n-1} m_k V\big(\theta_j(t) - \theta_k(t)\big) \quad \text{for } j = 0, \ldots, n-1. \tag{11}$$

To see this, we write $\delta_{\theta_j} = \delta_0(. - \theta_j) = \delta(. - \theta_j)$ and plug $f(t,\cdot) = \sum_j m_j \delta(. - \theta_j(t))$ into the transport equation $\partial_t f = \partial_\theta((V * f)f)$. For the left hand side we get

$$\frac{\partial f}{\partial t}(t,\cdot) = \sum_{j=0}^{n-1} m_j \left(-\frac{d\theta_j}{dt}(t)\right) \delta'(. - \theta_j(t)), \tag{12}$$

and for the right hand side

$$\begin{aligned} \partial_\theta\left((V * f(t,\cdot))f(t,\cdot)\right) &= \partial_\theta \left(\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} m_j m_k V\big(\theta_j(t) - \theta_k(t)\big)\, \delta(. - \theta_j(t)) \right) \\ &= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} m_j m_k V\big(\theta_j(t) - \theta_k(t)\big)\, \delta'(. - \theta_j(t)). \end{aligned} \tag{13}$$

Comparing (12) and (13) we deduce (11).

The case $n = 2$ is interesting. Since $V(0) = 0$ and V is odd, the system is

$$\begin{aligned} \dot{\theta}_0 &= -m_0 V(\theta_0 - \theta_1) \\ \dot{\theta}_1 &= -m_1 V(\theta_1 - \theta_0) = m_1 V(\theta_0 - \theta_1) \end{aligned}$$

hence (recall that $m_0 + m_1 = 1$)

$$\frac{d}{dt}(\theta_0 - \theta_1) = -V(\theta_0 - \theta_1). \tag{14}$$

Because $\theta_j(t) \to \bar{\theta}_j$ implies $\delta_{\theta_j(t)} \xrightarrow{\mathcal{D}} \delta_{\bar{\theta}_j}$, we may conclude the following.

Example 5.1. Let $0 \le \theta_v \le \frac{1}{2}$ and $V \in \mathcal{C}^1(S^1)$ odd with $V > 0$ on $]0, \theta_v[$ and $V < 0$ on $]\theta_v, \frac{1}{2}[$. If $\operatorname{dist}(\theta_0(0), \theta_1(0)) < \theta_v$, then $\theta_0(t) - \theta_1(t) \to 0$ for $t \to \infty$, i.e., the solution of (11) converges to a single peak; if $\operatorname{dist}(\theta_0(0), \theta_1(0)) > \theta_v$, then $\operatorname{dist}(\theta_0(t), \theta_1(t)) \to \frac{1}{2}$, hence the solution of (11) converges to two opposite peaks. $\operatorname{dist}(\theta_0, \theta_1) = \theta_v$ is an unstable stationary solution.

We are now in a good position to show that one gets into trouble when defining a 'first moment' in the 'obvious' naive way by $\int_{-\frac{1}{2}}^{\frac{1}{2}} p^*(f)(t,x)x\,dx$. The point is that this is (in general) *not* time-invariant.

Example 5.2. Let $0<\varepsilon<\frac{1}{8}$, V odd with $V(\theta)=\theta$ on $[0,4\varepsilon]$ and

$$f(0,\cdot)=\tfrac{1}{2}\delta_{\theta_0(0)}+\tfrac{1}{2}\delta_{\theta_1(0)}\quad\text{where}\quad \theta_0(0)=p(-\tfrac{1}{2}+\varepsilon)\quad\text{and}\quad \theta_1(0)=p(\tfrac{1}{2}-3\varepsilon).$$

Then

$$f(t,\cdot)\to\tfrac{1}{2}\left(\delta(.-(-\tfrac{1}{2}-\varepsilon))+\delta(.-(\tfrac{1}{2}-\varepsilon))\right)\overset{S^1}{=}\delta(.-(\tfrac{1}{2}-\varepsilon))\quad\text{as } t\to\infty.$$

To see this, note that
i) $\frac{d\theta_0}{dt}(0)=-\frac{1}{2}V(\theta_0(0)-\theta_1(0))=-\frac{1}{2}V(4\varepsilon)<0$ and $\frac{d\theta_1}{dt}(0)>0$; hence, $\mathrm{dist}(\theta_0(t),\theta_1(t))$ is decreasing in $t=0$;
ii) $\frac{d}{dt}(1+\theta_0(t)-\theta_1(t))\overset{(14)}{=}-V(1+\theta_0(t)-\theta_1(t))=-(1+\theta_0(t)-\theta_1(t))$ as long as $\mathrm{dist}(\theta_0(t),\theta_1(t))\leq 4\varepsilon$.
Since $\mathrm{dist}(\theta_0(0),\theta_1(0))=4\varepsilon$, i) and ii) imply that $\mathrm{dist}(\theta_0(t),\theta_1(t))=1+\theta_0(t)-\theta_1(t)\to 0$ as $t\to\infty$;
iii) $\frac{d}{dt}(\theta_0(t)+\theta_1(t))\overset{(11)}{=}0$, therefore, $\theta_0(t)+\theta_1(t)=-2\varepsilon \pmod 1$ for all $t\geq 0$. These facts imply that $\theta_0(t)\to-\frac{1}{2}-\varepsilon$ and $\theta_1(t)\to\frac{1}{2}-\varepsilon$.
The 'first moment' of the initial distribution is $\int_{-\frac{1}{2}}^{\frac{1}{2}}p^*(f)(0,x)x\,dx=-\varepsilon$. The 'first moment' of the limit is $\int_{-\frac{1}{2}}^{\frac{1}{2}}\delta(x-(\frac{1}{2}-\varepsilon))x\,dx=\frac{1}{2}-\varepsilon$. Therefore, this 'first moment' is not invariant. (In fact, it jumps by $\frac{1}{2}$ when one of the two peaks moves through the point $p(\frac{1}{2})\in S^1$.)

We will now analyze the linear stability *w.r.t. the peak ansatz* of two selected stationary solutions, namely peaks in *one* place, i.e., $\theta_j=\theta_0$ for all $0\leq j<n$, and peaks with *equal* masses at *equal distances*, i.e., $m_j=\frac{1}{n}$ and $\theta_0\in S^1$, $\theta_j=\theta_{j-1}+\frac{1}{n}$ for $1\leq j<n$. Obviously, both are stationary solutions of equation (11). The matrix of the linearization is

(15)

$$A=\begin{pmatrix} -\sum\limits_{k=0,k\neq 0}^{n-1}m_kV'(\theta_0-\theta_k) & m_1V'(\theta_0-\theta_1) & \dots & m_nV'(\theta_0-\theta_{n-1})\\ m_0V'(\theta_1-\theta_0) & -\sum\limits_{k=0,k\neq 1}^{n-1}m_kV'(\theta_1-\theta_k) & \dots & m_{n-1}V'(\theta_1-\theta_{n-1})\\ \vdots & & \ddots & \\ m_0V'(\theta_{n-1}-\theta_0) & m_1V'(\theta_{n-1}-\theta_1) & \dots & -\sum\limits_{k=0,k\neq n-1}^{n-1}m_kV'(\theta_{n-1}-\theta_k) \end{pmatrix}.$$

In both cases A has a clear structure such that the eigenvalues can be calculated explicitly (remember $\sum m_k=1$ for the first case; if $\theta_j-\theta_{j+1}=\frac{1}{n}$ and $m_j=\frac{1}{n}$, then A is a symmetric and cyclic matrix, because V' is even). The eigenvalues are

(16)

$$\lambda_j=\begin{cases} 0 & \text{if}\quad j=0\\ -V'(0) & \text{if}\quad 1\leq j<n,\ \text{assuming that } \theta_k=\theta_0 \text{ for all } k\\ \frac{1}{n}\sum_{k=1}^{n-1}V'(\frac{k}{n})(-1+\cos(2\pi\frac{jk}{n})) & \text{if}\quad 1\leq j<n,\ \text{assuming that } \theta_k=\theta_0+\frac{k}{n} \text{ for all } k\end{cases}$$

One eigenvalue is zero, because of the translational invariance of the system. Note that $\lambda_j=\lambda_{n-j}$.

Obviously, if for all $1\leq j\leq n-1$ the eigenvalues λ_j are negative, then a single peak is, resp. n peaks with equal masses and distances are stable w.r.t. the peak ansatz. For a single peak a necessary and sufficient condition for this stability is $V'(0)>0$. However, it is also clear that $V'(0)>0$ alone is not sufficient for a single peak to be stable e.g. with respect to continuous perturbations (recall the example $V=\sin(4\pi\theta)$ and propositon 2.6). Therefore, the following instability conditions are perhaps more interesting than the stability conditions. We tackle the question of stability with respect to non-peak like perturbations in the next section.

Theorem 5.3 (Instability conditions)**.**
i) A single peak is unstable if $V'(0)$ is negative.
ii) Let $n \geq 2$; n peaks with equal masses and distances are unstable if there exists $1 \leq j \leq n-1$ such that $\sum_{k=1}^{n-1} V'(\frac{k}{n})(-1+\cos(2\pi\frac{jk}{n})) > 0$ $(*)$.
For $n \in \{2,3,4\}$ a sufficient condition for $(*)$ to hold is $V'(\frac{1}{n}) < 0$ and for $n = 2,3$ this is also necessary.

Proof. If $V'(0) < 0$ or $(*)$ holds for some j, respectively, then at least one eigenvalue is positive; this implies instability with respect to the peak ansatz. However, for stability in any reasonable sense stability with respect to the peak ansatz is necessary.
If $n = 2$, then $\lambda_1 = \frac{1}{2}V'(\frac{1}{2})(-1+\cos(\pi)) = -V'(\frac{1}{2})$.
If $n = 3$, then $\lambda_2 = \lambda_1 = \frac{1}{3}(V'(\frac{1}{3})(-1+\cos(\frac{2\pi}{3})) + V'(\frac{2}{3})(-1+\cos(\frac{4\pi}{3})) = \frac{2}{3}V'(\frac{1}{3})(-1+\cos(\frac{2\pi}{3}))$.
Similarly, for $n = 4$ we have $\lambda_2 = -4V'(\frac{1}{4})$. Thus, for $n \in \{2,3,4\}$ there exists $\lambda_j > 0$ if $V'(\frac{1}{n}) < 0$ and for $n = 2,3$ this condition is also necessary. □

Example 5.4. Primi et al. [16] consider examples with $V(\theta) = \text{sign}(\alpha)\,\sin(2\pi\theta + \alpha\,\sin(2\pi\theta))$ and find that four-peak like solutions are not stable if $\alpha = \pm 1.2$. For $D = 0$ this is explained by Theorem 5.3, because for all $0 < |\alpha| < \pi$

$$V'(\tfrac{1}{4}) = \text{sign}(\alpha)\,\cos(2\pi\tfrac{1}{4} + \alpha\,\sin(2\pi\tfrac{1}{4}))\,(2\pi + \alpha 2\pi\cos(2\pi\tfrac{1}{4})) = -2\pi\,\text{sign}(\alpha)\,\sin(\alpha) < 0.$$

5.2. **Stability in the space of differentiable measures.**

We consider the linear stability of the stationary solution $f(t,\cdot) = \delta$ in the space of *differentiable measures* on S^1, $\mathcal{D}^1(S^1)$. Recall that this is the dual space of $\mathcal{C}^1(S^1)$ and can be identified with the subspace of distributions in $\mathcal{D}(S^1)$ of order at most 1. Note that δ_θ is close to δ in $\mathcal{D}^1(S^1)$ when θ is small (since $\langle \delta_\theta - \delta, h\rangle = \theta h'(\tilde\theta)$ for some $\tilde\theta$ between 0 and θ, so that $\|\delta_\theta - \delta\|_{\mathcal{D}^1} \leq |\theta|$).

We formulate a lemma that we will need later.

Lemma 5.5. *Let L be a (time-independent) differential operator on S^1, and let $\hat{L}$ be another differential operator on S^1 such that*

$$\langle Lf, h\rangle = \langle f, \hat{L}h\rangle$$

for $f \in \mathcal{D}(S^1)$ and $h \in \mathcal{C}^\infty(S^1)$. Let $f(t,\cdot) \in \mathcal{D}(S^1)$ be a solution of the PDE $\partial_t f = Lf$. Let $H(t,\cdot) \in \mathcal{C}^\infty(S^1)$ be the solution of the PDE $\partial_t H = \hat{L}H$ such that $H(0,\cdot) = h$. Then $\langle f(t,\cdot), H(-t,\cdot)\rangle$ is constant. In particular,

$$\langle f(t,\cdot), h\rangle = \langle f(0,\cdot), H(t,\cdot)\rangle.$$

Proof. We have

$$\begin{aligned}\frac{d}{dt}\langle f(t,\cdot), H(-t,\cdot)\rangle &= \langle \partial_t f(t,\cdot), H(-t,\cdot)\rangle + \langle f(t,\cdot), -(\partial_t H)(-t,\cdot)\rangle \\ &= \langle Lf(t,\cdot), H(-t,\cdot)\rangle + \langle f(t,\cdot), -\hat{L}H(-t,\cdot)\rangle \\ &= \langle f(t,\cdot), \hat{L}H(-t,\cdot) - \hat{L}H(-t,\cdot)\rangle = 0.\end{aligned}$$

Applying this with $\tilde{h} = H(t,\cdot)$ instead of h to obtain $\tilde{H}$, we have $\tilde{H}(-t,\cdot) = h$ and

$$\langle f(t,\cdot), h\rangle = \langle f(t,\cdot), \tilde{H}(-t,\cdot)\rangle = \langle f(0,\cdot), \tilde{H}(0,\cdot)\rangle = \langle f(0,\cdot), H(t,\cdot)\rangle.$$ □

Since the solution space of our equation is invariant with respect to translations, no stationary solution can be absolutely linearly stable. In order to deal with this technical problem, we will consider perturbations that do not change the barycenter.

If we set $f(t,\cdot) = \delta + \tilde{f}(t,\cdot)$ and linearize in equation (1) with $D = 0$, we obtain the linear PDE

$$\frac{\partial \tilde{f}}{\partial t}(t,\theta) = \frac{\partial}{\partial \theta}\big(V\tilde{f}(t,\cdot) + (V * \tilde{f}(t,\cdot))\delta\big)(\theta)\,. \tag{17}$$

A similar linearized equation has been used by Fellner and Raoul for the transport equation on $\mathbb{R}$. However, the proof of Theorem 5.6 below is completely different from the proof of their Theorem 3.1, since they work on disjoint intervals, which reduces the PDE to a finite-dimensional problem.

Theorem 5.6 (Linear stability of a single peak w.r.t. differentiable measures)**.** *Let $V \in \mathcal{C}^2(S^1)$ be odd and such that $V > 0$ on $]0, \frac{1}{2}[$ and $V'(0) > 0$. Assume that $\tilde{f}(t,\cdot) \in \mathcal{D}^1(S^1)$ is a solution of equation (17) such that $\operatorname{supp}\tilde{f}(0,\cdot) \subset I$ with a closed interval $0 \in I \subset S^1$ with $p(\frac{1}{2}) \notin I$. We can lift I uniquely to an interval $I' \subset \mathbb{R}$ with $0 \in I'$ and $p(I') = I$. We assume that $\langle \tilde{f}(0,\cdot), 1\rangle = \langle \tilde{f}(0,\cdot), \ell\rangle = 0$ where ℓ is a function on S^1 that satisfies $\ell(p(x)) = x$ for $x \in I'$. Then $\tilde{f}(t,\cdot)$ converges to zero as $t \to \infty$ in $\mathcal{D}^1(S^1)$.*

It is perhaps interesting to compare Theorem 4.2 with Theorem 5.6. The former shows that an initial distribution that is contained in an interval covering less than half of the circle will converge to a delta peak under equation (1) without diffusion. The latter shows that this peak is stable with respect to small perturbations that avoid an arbitrarily small neighborhood of the point opposite to the location of the peak.

Proof. Let $h \in \mathcal{C}^1(S^1)$. We have to show that $\langle \tilde{f}(t,\cdot), h\rangle \to 0$ as $t \to \infty$. We have

$$\begin{aligned}
\frac{d}{dt}\langle \tilde{f}(t,\cdot), h\rangle &= \Big\langle \frac{\partial \tilde{f}}{\partial t}(t,\cdot), h\Big\rangle \\
&= \Big\langle \frac{\partial}{\partial \theta}\big(V\tilde{f}(t,\cdot) + (V * \tilde{f}(t,\cdot))\delta\big), h\Big\rangle \\
&= \langle V\tilde{f}(t,\cdot) + (V * \tilde{f}(t,\cdot))\delta, -h'\rangle \\
&= -\langle \tilde{f}(t,\cdot), Vh'\rangle - (V * \tilde{f}(t,\cdot))(0)h'(0) \\
&= \langle \tilde{f}(t,\cdot), -V(h' - h'(0))\rangle\,.
\end{aligned}$$

(The last equality uses that V is odd.) We note that if h is constant, then $\langle \tilde{f}(t,\cdot), h\rangle$ is constant in time and that if $h = c\ell$ on I, then the same is true. Since $\langle \tilde{f}(0,\cdot), 1\rangle = \langle \tilde{f}(0,\cdot), \ell\rangle = 0$, $\langle \tilde{f}(t,\cdot), h\rangle = 0$ for such h. We can therefore restrict to functions h satisfying $h(0) = h'(0) = 0$. Let $H(t,\cdot)$ denote the (unique) solution of the initial value problem

$$\frac{\partial H}{\partial t} = -V\frac{\partial H}{\partial \theta}\,, \qquad H(0,\cdot) = h\,.$$

Then we see by Lemma 5.5 that $\langle \tilde{f}(t,\cdot), h\rangle = \langle \tilde{f}(0,\cdot), H(t,\cdot)\rangle$. (In particular, this shows that equation (17) has a unique solution in $\mathcal{D}^1(S^1)$ under the given assumptions.) Let $\Phi : \mathbb{R} \times S^1 \to S^1$ denote the flow associated to V, i.e.,

$$\frac{\partial \Phi}{\partial t}(t,\theta) = V\big(\Phi(t,\theta)\big)\,, \qquad \Phi(0,\theta) = \theta\,.$$

Then $H(t,\Phi(t,\theta)) = h(\theta)$, as can be readily checked. Equivalently, $H(t,\theta) = h(\Phi(-t,\theta))$. Now we claim that $H(t,\cdot)|_I$ converges to zero in $\mathcal{C}^1(I)$. For this, note first that for $\theta \in I$, we have

$\Phi(-t,\theta) \to 0$ as $t \to \infty$ uniformly in θ (this is because $p(\frac{1}{2})$ is the unique attracting and $p(0)$ the unique repelling fixed point of the flow Φ). So $\|H(t,\cdot)|_I\|_\infty \to |h(0)| = 0$. Next, we observe that

$$\frac{\partial}{\partial t}\frac{\partial}{\partial\theta}\Phi(-t,\theta) = -\frac{\partial}{\partial\theta}\frac{\partial\Phi}{\partial t}(-t,\theta) = -\frac{\partial}{\partial\theta}V\big(\Phi(-t,\theta)\big) = -V'\big(\Phi(-t,\theta)\big)\frac{\partial}{\partial\theta}\Phi(-t,\theta)\,.$$

For large t, $\Phi(-t,\theta)$ will be uniformly close to zero, so $-V'\big(\Phi(-t,\theta)\big)$ will be uniformly negative (recall that $V'(0) > 0$). This shows that $\frac{\partial}{\partial\theta}\Phi(-t,\theta)$ tends to zero as $t \to \infty$, uniformly for $\theta \in I$. This in turn implies that

$$\Big|\frac{\partial H}{\partial\theta}(t,\theta)\Big| = \Big|h'\big(\Phi(-t,\theta)\big)\frac{\partial}{\partial\theta}\Phi(-t,\theta)\Big|$$

also tends to zero uniformly on I as $t \to \infty$. So

$$\langle\tilde f(t,\cdot), h\rangle = \langle\tilde f(0,\cdot), H(t,\cdot)\rangle \to 0 \qquad \text{as } t \to \infty,$$

and this means that $\tilde f(t,\cdot) \to 0$ in $\mathcal{D}^1(S^1)$. More precisely, it follows that $\operatorname{supp}\tilde f(t,\cdot) \subset \Phi(-t,I)$, so that the support is contracted to $\{0\}$, whereas mass and first moment are always zero. ☐

It is certainly natural to consider perturbations that do not change the total mass (thinking of redistributing the mass on the circle). What about perturbations that do not preserve the barycenter? Consider a small perturbation g in $\mathcal{D}^1(S^1)$ with mass zero and $\langle g,\ell\rangle = M$ with $|M| \ll 1$. Then $\delta + g = \delta_M + (\delta - \delta_M + g)$, and $\delta - \delta_M + g$ is still a small perturbation, but now of δ_M. Assuming that $p(\frac{1}{2}) \notin I - M$, the theorem above then predicts convergence to the shifted peak δ_M.

In a way, we can see this from the proof. If we do not assume that $M = \langle\tilde f(0,\cdot),\ell\rangle = 0$, then (using test functions h with $h(0) = 0$, but not assuming $h'(0) = 0$) we find that

$$\tilde f(t,\cdot) \overset{\mathcal{D}}{\to} -M\delta'.$$

This is in accordance with $\delta_M - \delta \approx -M\delta'$.

Proposition 2.7 yields the following generalization for n equally distanced peaks with equal masses.

Corollary 5.7 (Stability of n peaks with respect to $\frac{1}{n}$-periodic perturbations)**.** *Let $n \geq 1$, let $V \in \mathcal{C}^2(S^1)$ be odd and such that $V_n > 0$ on $\left]0, \frac{1}{2n}\right[$ and $V_n'(0) > 0$. Assume that $\tilde f(t,\cdot) \in \mathcal{D}^1(S^1)$ is an $\frac{1}{n}$-periodic solution of equation (17) such that $\operatorname{supp}\tilde f(0,\cdot) \subset \bigcup_{j=0}^{n-1}(I + \frac{j}{n})$ with a closed interval $0 \in I \subset S^1$ with $p(\pm\frac{1}{2n}) \notin I$. We can lift I uniquely to an interval $I' \subset \mathbb{R}$ with $0 \in I'$ and $p(I') = I$. We assume that $\langle\tilde f(0,\cdot),1\rangle = \langle\tilde f(0,\cdot),\ell\rangle = 0$ where ℓ is a function on S^1 that satisfies $\ell(p(x)) = x$ for $x \in I'$ and $\ell(\theta) = 0$ for $\theta \in \bigcup_{j=1}^{n-1}(I + \frac{j}{n})$. Then $\tilde f(t,\cdot)$ converges to zero as $t \to \infty$ in $\mathcal{D}^1(S^1)$.*

We now want to derive a result similar to Theorem 5.6, but for two opposite peaks of not necessarily equal mass. We take this stationary solution to be $f_0 = m_-\delta_{-1/4} + m_+\delta_{1/4}$. If we set $f = f_0 + \tilde f$ in equation (1) with $D = 0$ and linearize, we obtain

$$\begin{aligned}\frac{\partial\tilde f}{\partial t}(t,\theta) = \frac{\partial}{\partial\theta}\Big(&\big(m_-V(\theta+\tfrac14) + m_+V(\theta-\tfrac14)\big)\tilde f(t,\theta) \\ &+ m_-(V*\tilde f(t,\cdot))(-\tfrac14)\delta_{-1/4}(\theta) + m_+(V*\tilde f(t,\cdot))(\tfrac14)\delta_{1/4}(\theta)\Big).\end{aligned} \tag{18}$$

We will define

$$\tilde V(\theta) = m_-V(\theta+\tfrac14) + m_+V(\theta-\tfrac14).$$

Theorem 5.8 (Linear stability of two peaks w.r.t. differentiable measures)**.** *Let $V \in \mathcal{C}^2(S^1)$ be odd and such that $V'(0) > 0$ and $V'(\frac{1}{2}) > 0$. With the notations m_-, m_+ and $\tilde V$ from above, suppose that $\tilde V$ has exactly four zeros on S^1, namely $-\frac{1}{4}$, θ_0, $\frac{1}{4}$ and θ_1 (in counter-clockwise order). Assume that $\tilde f(t,\cdot) \in \mathcal{D}^1(S^1)$ is a solution of equation (18) such that $\operatorname{supp} \tilde f(0,\cdot) \subset I_- \cup I_+$ with closed intervals $\pm\frac{1}{4} \in I_\pm \subset S^1$ such that $I_- \subset p\,(]\theta_1 - 1, \theta_0[)$ and $I_+ \subset p\,(]\theta_0, \theta_1[)$. Let $\ell \in \mathcal{C}^\infty(S^1)$ be such that $\ell(\theta) = \theta - (\pm\frac{1}{4})$ for $\theta \in I_\pm$. We assume that $m(I_+, \tilde f(0,\cdot)) = m(I_-, \tilde f(0,\cdot)) = \langle \tilde f(0,\cdot), \ell\rangle = 0$. Then $\tilde f(t,\cdot)$ converges to zero as $t \to \infty$ in $\mathcal{D}^1(S^1)$.*

Note that $\tilde V$ has to have at least four zeros, since $\tilde V'$ is positive at the two zeros at $\pm\frac{1}{4}$.

Note also that because of the translational invariance, any two peaks with distance $\frac{1}{2}$ are stable under the assumptions of Theorem 5.8.

Proof. We proceed in a similar way as in the proof of Theorem 5.6. We find that

$$\frac{\partial}{\partial t}\langle \tilde f(t,\cdot), h\rangle = -\big\langle \tilde f(t,\cdot), m_- V(\cdot + \tfrac{1}{4})(h' - h'(-\tfrac{1}{4})) + m_+ V(\cdot - \tfrac{1}{4})(h' - h'(\tfrac{1}{4}))\big\rangle\,.$$

So we let $H(t,\theta)$ be the solution of

$$\frac{\partial H}{\partial t}(t,\theta) = -m_- V(\theta + \tfrac{1}{4})\Big(\frac{\partial H}{\partial \theta}(t,\theta) - \frac{\partial H}{\partial \theta}(t, -\tfrac{1}{4})\Big) - m_+ V(\theta - \tfrac{1}{4})\Big(\frac{\partial H}{\partial \theta}(t,\theta) - \frac{\partial H}{\partial \theta}(t, \tfrac{1}{4})\Big)$$

with $H(0,\cdot) = h$; then $\langle \tilde f(t,\cdot), h\rangle = \langle \tilde f(0,\cdot), H(t,\cdot)\rangle$ by Lemma 5.5, and we have to figure out the long-term behavior of H. We see that a function h that is constant separately on I_- and on I_+ is a stationary solution on $I_- \cup I_+$ and that the same is true when h is a multiple of ℓ. So we can assume that $h(\frac{1}{4}) = h(-\frac{1}{4}) = m_+ h'(\frac{1}{4}) + m_- h'(-\frac{1}{4}) = 0$. We write H' for $\frac{\partial H}{\partial \theta}$. Then we have that

$$\frac{d}{dt} H'(t, \tfrac{1}{4}) = -m_- V'(\tfrac{1}{2})\big(H'(t, \tfrac{1}{4}) - H'(t, -\tfrac{1}{4})\big)$$

and

$$\frac{d}{dt} H'(t, -\tfrac{1}{4}) = -m_+ V'(\tfrac{1}{2})\big(H'(t, -\tfrac{1}{4}) - H'(t, \tfrac{1}{4})\big).$$

This shows that $m_+ H'(t, \frac{1}{4}) + m_- H'(t, -\frac{1}{4}) = 0$ for all t and that $H'(t, \frac{1}{4}) - H'(t, -\frac{1}{4}) \to 0$ as $t \to \infty$ (recall that $V'(\frac{1}{2}) > 0$). So $H'(t, \frac{1}{4}) \to 0$ and $H'(t, -\frac{1}{4}) \to 0$. By arguments similar to those in the proof of Theorem 5.6 (note that in the present situation, the flow associated to $\tilde V$ moves the values of h away from $\frac{1}{4}$ and $-\frac{1}{4}$ and toward θ_0 and θ_1), we then see that $H(t,\cdot) \to 0$ in $\mathcal{C}^1(I_- \cup I_+)$ and therefore $\langle \tilde f(t,\cdot), h\rangle \to 0$ as $t \to \infty$. For a general test function h, we then find that

$$H(t,\cdot) \to h(\tfrac{1}{4})\chi_+ + h(-\tfrac{1}{4})\chi_- + (m_+ h'(\tfrac{1}{4}) + m_- h'(-\tfrac{1}{4}))\ell \quad \text{on } I_- \cup I_+,$$

where $\chi_\pm$ is a function in $\mathcal{C}^\infty(S^1)$ that takes the value 1 on $I_\pm$ and the value 0 on $I_\mp$. This translates into

$$\tilde f(t,\cdot) \xrightarrow{\mathcal{D}} m(I_+, \tilde f(0,\cdot))\delta_{1/4} + m(I_-, \tilde f(0,\cdot))\delta_{-1/4} - \langle \tilde f(0,\cdot), \ell\rangle (m_+ \delta'_{1/4} + m_- \delta'_{-1/4}) = 0. \qquad \square$$

The need for the three assumptions $m(I_+, \tilde f(0,\cdot)) = m(I_-, \tilde f(0,\cdot)) = \langle \tilde f(0,\cdot), \ell\rangle = 0$ arises because two opposite peaks of arbitrary masses and arbitrary orientation form a stationary solution. If we have a perturbation that violates these assumptions (but does not change the total mass), say

$$m(I_+, \tilde f(0,\cdot)) = \mu, \quad m(I_-, \tilde f(0,\cdot)) = -\mu \quad \text{and} \quad \langle \tilde f(0,\cdot), \ell\rangle = M,$$

then we can proceed as in the one-peak case. We adjust masses and orientation to obtain

$$(m_+ + \mu)\delta_{1/4+M} + (m_- - \mu)\delta_{-1/4+M}$$

as a stationary solution such that the resulting perturbation of this solution satisfies the assumptions.

If there is some $0 < \theta_v < \frac{1}{2}$ such that $V(\theta_v) = 0$ and $V'(\theta_v) > 0$ (and $V'(0) > 0$, of course), then we expect two peaks at a distance of θ_v also to be a stable stationary solution, up to a redistribution of mass between the two peaks and reorientation that preserves the distance. This is indeed the case.

Corollary 5.9. *Let $V \in \mathcal{C}^2(S^1)$ be odd and such that $V'(0) > 0$, and assume that there is $0 < \theta_v < \frac{1}{2}$ such that $V(\theta_v) = 0$ and $V'(\theta_v) > 0$. Let $m_\pm > 0$ with $m_+ + m_- = 1$, and consider the stationary solution $f_0 = m_+\delta_{\theta_v/2} + m_-\delta_{-\theta_v/2}$ of equation (1) with $D = 0$. Let $\tilde{V}(\theta) = m_-V(\theta + \frac{\theta_v}{2}) + m_+V(\theta - \frac{\theta_v}{2})$ and suppose that $\tilde{V}$ has exactly four zeros on S^1, namely $-\frac{\theta_v}{2}$, θ_0, $\frac{\theta_v}{2}$ and θ_1 (in counter-clockwise order). Then f_0 is linearly stable with respect to perturbations satisfying the conditions in Theorem 5.8 (with $\pm\frac{1}{4}$ replaced by $\pm\frac{\theta_v}{2}$).*

Proof. The proof is virtually identical to the proof of Theorem 5.8, after replacing $\pm\frac{1}{4}$ by $\pm\frac{\theta_v}{2}$. □

We saw that single peaks are stable up to reorientation if $V'(0) > 0$. Two opposite peaks are stable up to redistribution of mass and reorientation preserving the distance under the assumptions of Theorem 5.8, which include $V'(0) > 0$ and $V'(\frac{1}{2}) > 0$. Now we prove that $n \geq 3$ equal peaks at equal distances are stable if $V'(\frac{j}{n}) > 0$ for $0 \leq j \leq n-1$, up to ? — we shall see.

Theorem 5.10. *Let $n \geq 3$. Let $V \in \mathcal{C}^2(S^1)$ be odd and such that $V'(\frac{j}{n}) > 0$ for all $0 \leq j < n$ and $V_n > 0$ on $\left]0, \frac{1}{2n}\right[$. Let, for $0 \leq j < n$, I_j be a closed interval in S^1 contained in $\left]\frac{j}{n} - \frac{1}{2n}, \frac{j}{n} + \frac{1}{2n}\right[$, and let $\ell \in \mathcal{C}^\infty(S^1)$ be such that $\ell(\theta) = \theta - \frac{j}{n}$ for $\theta \in I_j$, for all j. Then*

$$\text{the stationary solution} \qquad f_0 = \frac{1}{n}\sum_{j=0}^{n-1}\delta_{j/n} \qquad \text{of equation (1) with } D = 0$$

is linearly stable with respect to perturbations $\tilde{f} \in \mathcal{D}^1(S^1)$ such that

$$\operatorname{supp}\tilde{f} \subset \bigcup_{j=0}^{n-1} I_j, \qquad m(I_j, \tilde{f}) = 0 \quad \text{for all } 0 \leq j < n, \qquad \text{and} \qquad \langle \tilde{f}, \ell\rangle = 0.$$

Proof. The proof proceeds in a way analogous to the proofs of Theorems 5.6 and 5.8. The equation governing the development of $H(t,\cdot)$ is (writing again H' for $\frac{\partial H}{\partial\theta}$)

$$\frac{\partial H}{\partial t}(t,\theta) = -\frac{1}{n}\sum_{j=0}^{n-1} V(\theta - \tfrac{j}{n})\big(H'(t,\theta) - H'(t,\tfrac{j}{n})\big). \tag{19}$$

The flow associated to $V_n = \sum_j V(\cdot - \frac{j}{n})$ moves away from the points $\frac{j}{n}$ toward the points $\frac{j}{n} \pm \frac{1}{2n}$. So for any test function h satisfying $h(\frac{j}{n}) = h'(\frac{j}{n}) = 0$ for all j, we find that $\langle \tilde{f}(t,\cdot), h\rangle \to 0$ as $t \to \infty$ in the same way as before. For the derivatives $H'(t, \frac{j}{n})$ we obtain the equation (using $V_n(\frac{j}{n}) = 0$)

$$\frac{d}{dt}H'\Big(t, \frac{j}{n}\Big) = -\frac{1}{n}\sum_{k=0}^{n-1} V'\Big(\frac{j-k}{n}\Big)\Big(H'\Big(t,\frac{j}{n}\Big) - H'\Big(t,\frac{k}{n}\Big)\Big).$$

This leads to

$$\frac{d}{dt}\sum_{j=0}^{n-1} H'\Big(t,\frac{j}{n}\Big)^2 = -\frac{1}{2n}\sum_{j,k=0}^{n-1} V'\Big(\frac{j-k}{n}\Big)\Big(H'\Big(t,\frac{j}{n}\Big) - H'\Big(t,\frac{k}{n}\Big)\Big)^2 \leq 0,$$

with equality only if all $H'(t, \frac{j}{n})$ are equal. On the other hand, one sees easily that $\sum_j H'(t, \frac{j}{n})$ is constant. Together, this implies that all $H'(t, \frac{j}{n})$ converge to the same value as $t \to \infty$. Since functions that are constant on each I_j and also ℓ are stationary under equation (19), we get that

$$H(t, \cdot) \to \sum_{j=0}^{n-1} h\Big(\frac{j}{n}\Big)\chi_j + \frac{1}{n}\sum_{j=0}^{n-1} h'\Big(\frac{j}{n}\Big)\ell \qquad \text{on } \bigcup_{j=0}^{n-1} I_j,$$

where $\chi_j \in \mathcal{C}^\infty(S^1)$ is a function that takes the value 1 on I_j and the value 0 on all I_k with $k \neq j$. In terms of $\tilde{f}$, this reads

$$\tilde{f}(t, \cdot) \xrightarrow{\mathcal{D}} \sum_{j=0}^{n-1} m(I_j, \tilde{f}(0, \cdot))\delta_{j/n} - \langle \tilde{f}(0, \cdot), \ell\rangle \frac{1}{n}\sum_{j=0}^{n-1} \delta'_{j/n} = 0. \qquad \square$$

As before, if $M = \langle \tilde{f}, \ell\rangle \neq 0$, then we expect a reorientation by M in the positive direction. It is less clear what happens when mass is redistributed between the domains of attraction of the various peaks. The proof above would suggest that we simply end up with equidistant peaks of different masses, but this will in general no longer be a stationary solution. If we consider the system of ODEs (11) for moving peaks, then we see by the theorem on implicit functions that there is a unique stationary solution up to translation near f_0 with peaks of prescribed slightly different masses if the matrix in (15) with $\theta_j = \frac{j}{n}$ and $m_j = \frac{1}{n}$ only has one vanishing eigenvalue — it is the matrix obtained as the Jacobian with respect to the θ_j of the map

$$(m_0, \dots, m_{n-1}; \theta_0, \dots, \theta_{n-1}) \longmapsto \Big(-\sum_{k=0}^{n-1} m_k V(\theta_j - \theta_k)\Big)_{0 \le j < n},$$

and the zero eigenvalue corresponds to an overall translation. This condition will be satisfied when the positions of n equidistant peaks of the same mass are stable up to translation, since then all the relevant eigenvalues are negative. Note that the condition on V in Theorem 5.10 is sufficient to ensure this is the case, compare the eigenvalues in equation (16) before Theorem 5.3.

In the special case that we have $V(\frac{j}{n}) = 0$ for all $0 \le j < n$ the stationary solution of the system (11) will consist of equidistant peaks even when the masses are not equal. In this case, one can formulate a variant of Theorem 5.10 in analogy to Theorem 5.8 that shows that n equidistant peaks with different masses are linearly stable with respect to perturbations respecting the mass distribution and the overall orientation.

6. Numerical algorithms and simulations

6.1. **Solving the transport-diffusion equation via the Fourier transformed system.**

In Section 6.3 we will calculate numerically solutions of the transport-diffusion equation (1) for randomly chosen as well as pre-structured initial distributions.

By using the Fourier transform we convert the partial differential equation into an infinite (but discrete) system of ordinary differential equations; since large Fourier coefficients of a smooth function are small, we can then restrict to a finite system, which can be solved very efficiently.

In Section 2.1 we found that the Fourier transform of the transport-diffusion equation (1) is given by (compare (2))

$$\begin{aligned} \dot{f}_k &= -(2\pi)^2 k^2 D f_k + 2\pi i k \sum_{l \in Z} V_l f_l f_{k-l} \\ &= c_k f_k + 4\pi k \sum_{l \in \mathbb{Z}, l \neq 0, k} v_l f_l f_{k-l} \quad \text{for } k \in \mathbb{Z}_{>0}, \end{aligned} \tag{20}$$

where the eigenvalues c_k of the system (see (3)) and $v_k \in \mathbb{R}$ are

$$c_k = -(2\pi)^2 k^2 D + 4\pi k v_k \qquad \text{and} \qquad v_k = \int_0^{\frac{1}{2}} V(\theta) \sin(2\pi k\theta)\, d\theta.$$

Mass conservation is reflected by $\dot{f}_0 = 0$; we put $f_0 = 1$. Note that the number of positive eigenvalues is usually small (see remarks 3.1).

In order to avoid the necessity to use very small timesteps (k^2 is large for higher modes) we multiply (20) by $\exp(-c_k t)$ and define $g_k(t) = f_k(t)\exp(-c_k t)$. Then we get

$$\dot{g_k}(t) = (\dot{f}_k(t) - c_k f_k(t)) e^{-c_k t} = 4\pi k e^{-c_k t} \sum_{l \neq 0, k} v_l f_l(t) f_{k-l}(t), \tag{21}$$

which we solve by a second-order scheme.

The number n of equations is adapted dynamically, in the following way. We start with n Fourier coefficients of f, assuming that higher modes are zero; we calculate the right hand side of (21) for $1 \le k \le 2n$ and accept for $1 \le k \le n$ the resulting $f_k(t + \Delta t)$ as the new value for f_k. If for some $\tilde{k} > n$ the slope of $f_{\tilde{k}}$ is larger than some (small) error bound, then the number n of equations is increased to $\tilde{k}+1$. The additionally needed Fourier coefficients f_k with $n < k \le \tilde{k}+1$ are initialized as zero.

A further advantage of this scheme is that higher periodicity of an initial function is preserved.

6.2. **Solving the stationary equation via iteration.**

We also programmed the iteration scheme which Primi et al. [16] used to prove existence of peak-like solutions.

We start with an arbitrary function $f^{(0)}$ on S^1 with given mass (usually 1). E.g., $f^{(0)}$ may be the solution of $D\frac{df^{(0)}}{d\theta}(\theta) = -V(\theta) f^{(0)}(\theta)$, which is expected to lie near the one-peak solution, if it exists (see Primi et al. [16]; $V = V * \delta \approx V * f$ if f is one-peak like).

Then we iterate

$$D\frac{df^{(n+1)}}{d\theta}(\theta) = -(V * f^{(n)})(\theta)\, f^{(n+1)}(\theta) \quad \text{and require} \quad \int_{S^1} f^{(n+1)}(\theta)\, d\theta = \int_{S^1} f^{(0)}(\theta)\, d\theta,$$

so that $f^{(n+1)}$ is a function on S^1 with the same mass as $f^{(0)}$.

If this sequence converges, then the limit is obviously a solution of (4), i.e., it is a stationary solution of (1). Primi et al. [16] give criteria for convergence; e.g., the assumptions $V'(0) > 0$ and $\int_0^\theta V(\psi)\, d\psi > 0$ for $\theta \in \left]0, \frac{1}{2}\right]$ imply the existence of one-peak like solutions if D is small.

Our iteration program reliably finds stationary solutions with $\frac{1}{n}$-periodicity if no stationary solutions with lower periodicity are present. Otherwise, it is better to use $\tilde{V}_n$ and $n^2 D$ instead of V and D, compare Proposition 2.7. (Unfortunately, in our program numerical instabilities accumulate, so that $\frac{1}{n}$-periodicity of $f^{(0)}$ for $n > 1$ is not preserved numerically, in contrast to the theoretical prediction.)

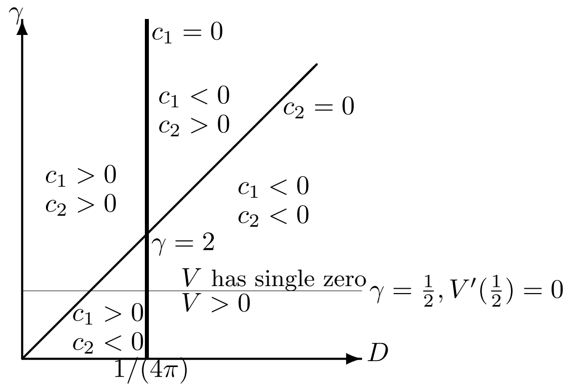

FIGURE 2. Regions of in/stability of first and second eigenvalue

6.3. **Examples.**

In the following examples the stationary solutions were calculated with both algorithms (exceptions will be mentioned); their stability was checked with the Fourier based system. (An upper horizontal line in the figures is the homogeneous solution 1.)

The first example is interesting because it shows a backward bifurcation and mixed mode solutions.

Example 6.1. Let $D > 0$, $\gamma \geq 0$ and define

$$V(\theta) = \sin(2\pi\theta) + \gamma \sin(4\pi\theta).$$

We use formula (3) for the eigenvalues and get

$$c_1 = -4\pi^2 D + 4\pi \int_0^{\frac{1}{2}} (\sin^2(2\pi\theta) + \gamma \sin(4\pi\theta)\sin(2\pi\theta))\, d\theta = \pi\,(-4\pi D + 1) > 0 \iff D < \frac{1}{4\pi}$$

and

$$c_2 = -16\pi^2 D + 8\pi \int_0^{\frac{1}{2}} (\sin(2\pi\theta)\sin(4\pi\theta) + \gamma \sin^2(4\pi\theta))\, d\theta = 2\pi(-8\pi D + \gamma) > 0 \iff D < \frac{\gamma}{8\pi}.$$

For $(D, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+$ all four combinations of $c_1 <> 0$, $c_2 <> 0$ are possible, see Figure 2. All other eigenvalues are negative.

For all parameter values we have $V'(0) > 0$ and $\int_0^\theta V(\psi)\, d\psi > 0$ for $0 < \theta < \frac{1}{2}$; therefore for very small diffusion coefficient one-peak like solutions exist (Primi et al. [16]) and at least for $D = 0$ they are stable by Theorem 5.6. We find that $V_2(\theta) = \gamma \sin(4\pi\theta)$, therefore $V_2'(0) > 0$ and $\int_0^\theta V_2(\psi) d\psi > 0$ on $\left]0, \frac{1}{4}\right[$, thus two-peaks like solutions exist for small enough diffusion (Primi et al. [16]). For $\gamma < \frac{1}{2}$ we have $V'(0) > 0$ and $V'(\frac{1}{2}) < 0$, therefore two peaks are not stable, see Theorem 5.3; two-peaks like solutions are only stable within the space of $\frac{1}{2}$-periodic solutions (if D is sufficiently small), see Proposition 2.7 and Theorem 5.6 for $D = 0$; note that $c_2 > 0$. For $\gamma > \frac{1}{2}$, V has a single simple zero and $V'(\frac{1}{2}) > 0$, so for $D = 0$ two-peaks solutions are stable by Theorem 5.8.

For $D = \frac{1}{4\pi} \approx 0.0796$ and $\gamma = 2$, first and second eigenvalue of (1) are zero simultaneously. Therefore stationary solutions with two maxima of different height can be expected to exist, so-called 'mixed mode solutions' (Golubitsky and Schaeffer [10]); Figure 3 (top left figure) shows such solutions. Near that parameter combination there exist backward bifurcations which are (to our experience) unusual for the differential equation (1) on S^1. Interestingly, Chayes and Panferov [3] proved that on tori in dimension $d \geq 2$ *only* backward bifurcations are possible.

Figure 3 shows typical stationary solutions and their stability for $\gamma = 2$ and $\gamma = 4$. The $\frac{1}{2}$-periodic stationary solutions are unstable ($\gamma = 2$), or they become unstable with decreasing D ($\gamma = 4$). This suggests that in general, the stability result for two peaks at $D = 0$ cannot be carried over to (very small) $D > 0$. However, solutions need much longer times at smaller D to move beyond states with two peaks of different height. E.g., for D of size of the order of 0.05, $\gamma = 2$, and starting with a small perturbation of $f = 1$, we get two peaks of different heights in the first two time units, while convergence to the mixed-mode solution needs about 30 time units; for $D \approx 0.01$ and $\gamma = 2$ as well as $\gamma = 4$, these time scales change to one unit for initial pattern formation and several hundred units for convergence to one peak.

In Figure 3 the stationary solutions were generated with the iteration method, and their stability was tested with the Fourier algorithm. The unstable $\frac{1}{2}$-periodic steady states in Figure 3 are stable in the space of $\frac{1}{2}$-periodic functions; they are also found with the Fourier-based algorithm if the simulation is started with a $\frac{1}{2}$-periodic function.

The second example shows non-trivial solutions when $V'(0)$ and $V'(\frac{1}{2})$ are negative, and hence for zero diffusion coefficient neither one peak nor two peaks at distance $\frac{1}{2}$ are stable by Theorem 5.3.

Example 6.2. Let

$$V(\theta) = \sin(2\pi\theta) - \sin(4\pi\theta).$$

Only the first eigenvalue is positive for small enough diffusion coefficient. Figure 4 shows how the stationary solution is approached. As one expects according to Corollary 5.9 (which holds for $D = 0$), for small diffusion coefficient it consists of two peaks with distance θ_v (where $V(\theta_v) = 0$). We checked numerically that indeed $\tilde{V}'(0) = \frac{1}{2}(V'(0) + V'(\theta_v)) > 0$.

This solution could be calculated only with the Fourier transformed system, since the iteration method does not converge: it runs into a two-cycle.

The next example shows that one-peak and two-peaks like solutions are possible in the same model at the same parameter values. It is interesting that there exists a one-peak like solution although the first eigenvalue (of the linearization near the homogeneous solution) is negative for all parameter values.

Example 6.3. Let

$$V(\theta) = \sin(4\pi\theta) + \gamma \sin(6\pi\theta).$$

For $-\frac{2}{3} < \gamma < \frac{2}{3}$ the turning rate V has a single zero on $]0, \frac{1}{2}[$, $V'(0) > 0$ and $V'(\frac{1}{2}) > 0$; also, $V'(\frac{1}{3}) > 0$ for $\gamma > \frac{1}{3}$. If $\gamma > 0$, then for all $\delta > 0$ there is an $\varepsilon > 0$ such that $\int_0^\theta V(\psi)\, d\psi > \varepsilon$ for $\delta < \theta \leq \frac{1}{2}$. Theorem 7.3. in Primi et al. [16] shows that a one-peak like solution exists for small enough D. The second eigenvalue c_2 is positive for $D < \frac{1}{8\pi}$, the third eigenvalue c_3 is positive for $D < \frac{\gamma}{12\pi}$, which is ≈ 0.013 for $\gamma = 0.5$.

Figure 5 shows stationary solutions (left side; calculated with the iteration scheme) and how they are approached in time (right side; Fourier based program). We see that one-peak and two-peaks like solutions are locally stable for small enough D and $\gamma = 0.5$. The one-peak like solution develops when the initial distribution is sufficiently centered (compare Theorem 4.2), but in the simulations $f(0, \cdot)$ did not have compact support. The three-peaks solution is stable in the subspace of $\frac{1}{3}$-periodic functions; it is unstable for $D = 0.01$ (data not shown); For $D = 0.001$ a solution with three peaks of different height, of which only two had distance $\frac{1}{3}$, developed when the simulation was started with a perturbed three-peaks like function. Note that $V_{(3)}$ actually satisfies the 3-peaks stability conditions of Theorem 5.10.

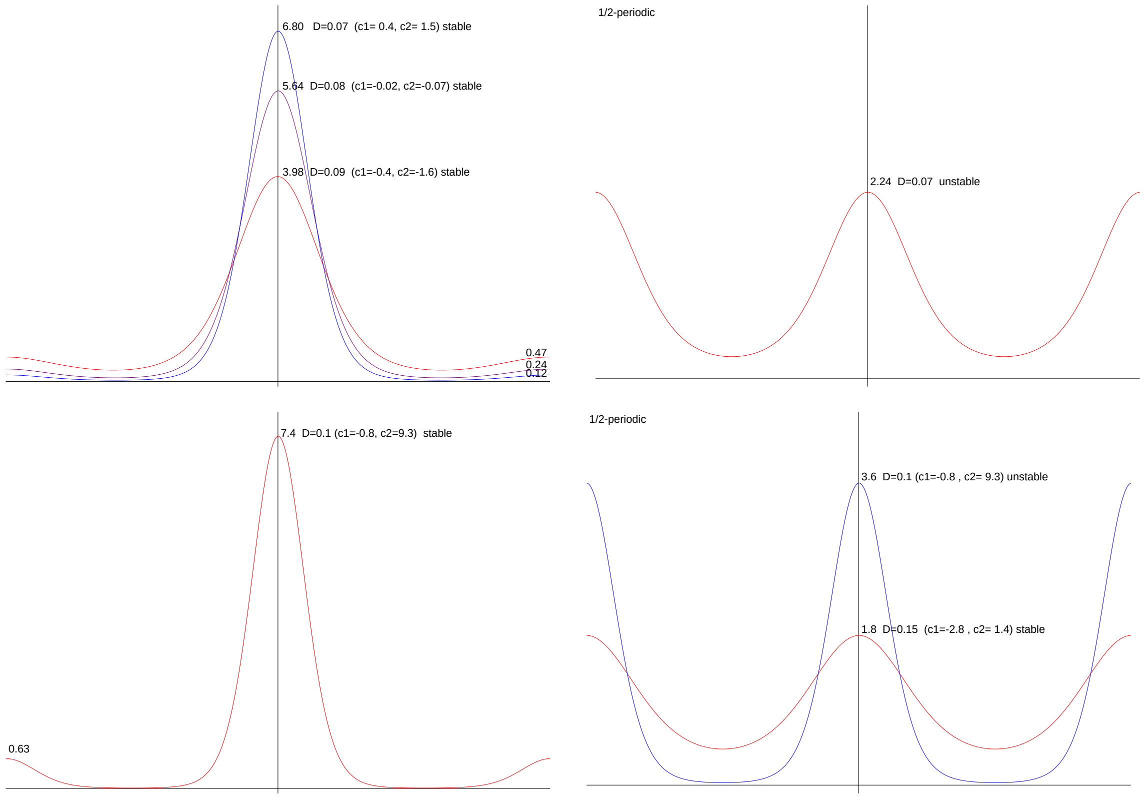

Figure 3. Stationary solutions for $V(\theta) = \sin(2\pi\theta) + \gamma \sin(4\pi\theta)$ with $\gamma = 2$ (top) and $\gamma = 4$ (bottom) ($V'(0) > 0, V'(\frac{1}{2}) > 0, V(0.29) = 0$ and $V(0.27) = 0$, respectively). (top left) Stable mixed mode solutions; the distance between the maxima is $\frac{1}{2}$. Note the backward bifurcation: there is a stable stationary solution even though both eigenvalues c_1 and c_2 are negative! For $D \geq 0.093$ and $\gamma = 2$ we found no non-constant stationary solution. (top right) A $\frac{1}{2}$-periodic stationary solution; it is unstable, but stable in the space of $\frac{1}{2}$-periodic solutions. For $D \geq \frac{1}{4\pi}$ there are no non-trivial $\frac{1}{2}$-periodic stationary solutions. (bottom left) Mixed mode solution with distance $\frac{1}{2}$ between the two maxima; for $D = 0.15$ we found no stationary mixed mode solution. (bottom right) $\frac{1}{2}$-periodic solutions. In mixed mode solutions the larger maximum grows with decreasing D while the second maximum vanishes. In $\frac{1}{2}$-periodic solutions the maxima grow with decreasing D. In all cases the peaks become narrower.

With small diffusion coefficient the ‘typical’ outcome of a simulation that is started with small deviations from $f = 1$ are one large and one small peak that are opposite. We suppose that for $D > 0$ these become equally high for large times; the smaller D is, the more time will be needed for that.

Caption for Figure 5. Top row (left): These stationary solutions for various D-values were calculated with the iteration algorithm; for $D >\approx 0.02$ the solutions look $\frac{1}{2}$-periodic; (right): A one-peak like solution with a small second maximum develops fast when the simulation is started with a centered distribution; here we started with the stationary solution for $V(\theta) = \sin(2\pi\theta)$, $D = 0.03$.

Second row (left): These $\frac{1}{2}$-periodic stationary solutions for various D-values were calculated with

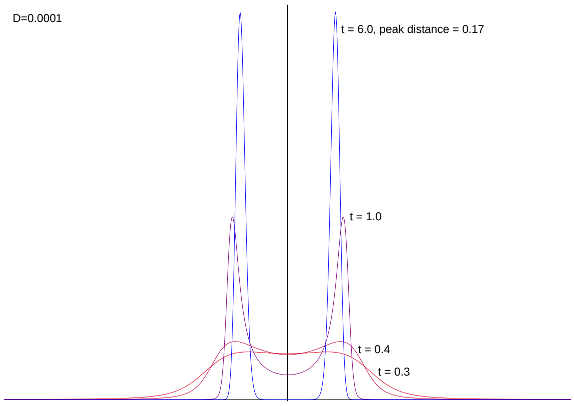

FIGURE 4. Development of the stationary solution for $V(\theta) = \sin(2\pi\theta) - \sin(4\pi\theta)$ started with $f(\theta, 0) = 1 + 0.8 \cos(2\pi\theta)$. $\theta_v = 0.17$ is the only non-trivial zero of V. The numerical computation started with 20 Fourier coefficients and ended with 140.

the iteration algorithm with forced $\frac{1}{2}$-periodicity. (right) Approximation in time of a $\frac{1}{2}$-periodic stationary solution. The simulation was started with $f(0, \theta) = \phi \sin(2\pi\theta)$, $\phi = 0.1$ (start not shown). Until $t \approx 8$ the distribution converges towards the homogeneous distribution (this is what it has to do: any initial distribution that is orthogonal to the modes occurring in V dies out), then triggered by some numerical noise, instability of the constant solution takes over, and the second mode begins to grow.
Third row (left): $\frac{1}{3}$-periodic solutions were calculated with the iteration scheme with forced $\frac{1}{3}$-periodicity. (right) The simulation was started with a small perturbation ($\operatorname{Re} f_1(0) = 0.01$, $\operatorname{Re} f_2(0) = 0.005$) of the $\frac{1}{3}$-periodic solution for $D = 0.01$. A distribution with three slightly different peaks develops very fast, where the distances between first/second and first/third peak are *not* $\frac{1}{3}$; then no further changes are discernible.
Bottom row: 'Typical' result of a simulation, here started with the $\frac{1}{3}$-periodic solution for $D = 0.01$ which was perturbed by $\operatorname{Re} f_1(0) = -0.01$, $\operatorname{Re} f_2(0) = -0.005$. A distribution with two different maxima at distance $\frac{1}{2}$ develops fast, then no further changes are discernible.

The last example shows that a variety of behaviors is possible if $V < 0$ on $\left]0, \frac{1}{2}\right[$.

Example 6.4. We compare

$$V_{(2)}(\theta) = \sin(4\pi\theta) - \gamma \sin(2\pi\theta) \quad (\gamma > 2) \qquad \text{and} \qquad V_{(3)}(\theta) = \sin(6\pi\theta) - \gamma \sin(2\pi\theta) \quad (\gamma > 3).$$

In both cases $V < 0$ on $\left]0, \frac{1}{2}\right[$, $V'(0) < 0$ and $V'(\frac{1}{2}) > 0$. For $V_{(2)}$ only the second eigenvalue is positive for $D < \frac{1}{8\pi}$; for $V_{(3)}$ only the third eigenvalue is positive for $D < \frac{1}{12\pi}$; all other eigenvalues are negative. We have $V_{(2),2} = 2 \sin(4\pi\theta) > 0$ on $\left]0, \frac{1}{4}\right[$ and $V_{(3),3} = 3 \sin(6\pi\theta) > 0$ on $\left]0, \frac{1}{6}\right[$; all other $V_{(j),n}$ are zero.

Therefore we expect (at small enough diffusion coefficient D) for $V_{(2)}$ stationary solutions with two equal maxima at distance $\frac{1}{2}$, and for $V_{(3)}$ three equal maxima with distance $\frac{1}{3}$. These develop indeed, but the time scales are interesting, see Figure 6. Two, resp. three different maxima develop very quickly but at unexpected distances; development toward equal distances and heights can be a very slow process. The explanation is that for both V there are orbits of other stationary solutions when $D = 0$: For $V_{(2)}$ two peaks whose masses add to 1; for $V_{(3)}$ three peaks whose positions and

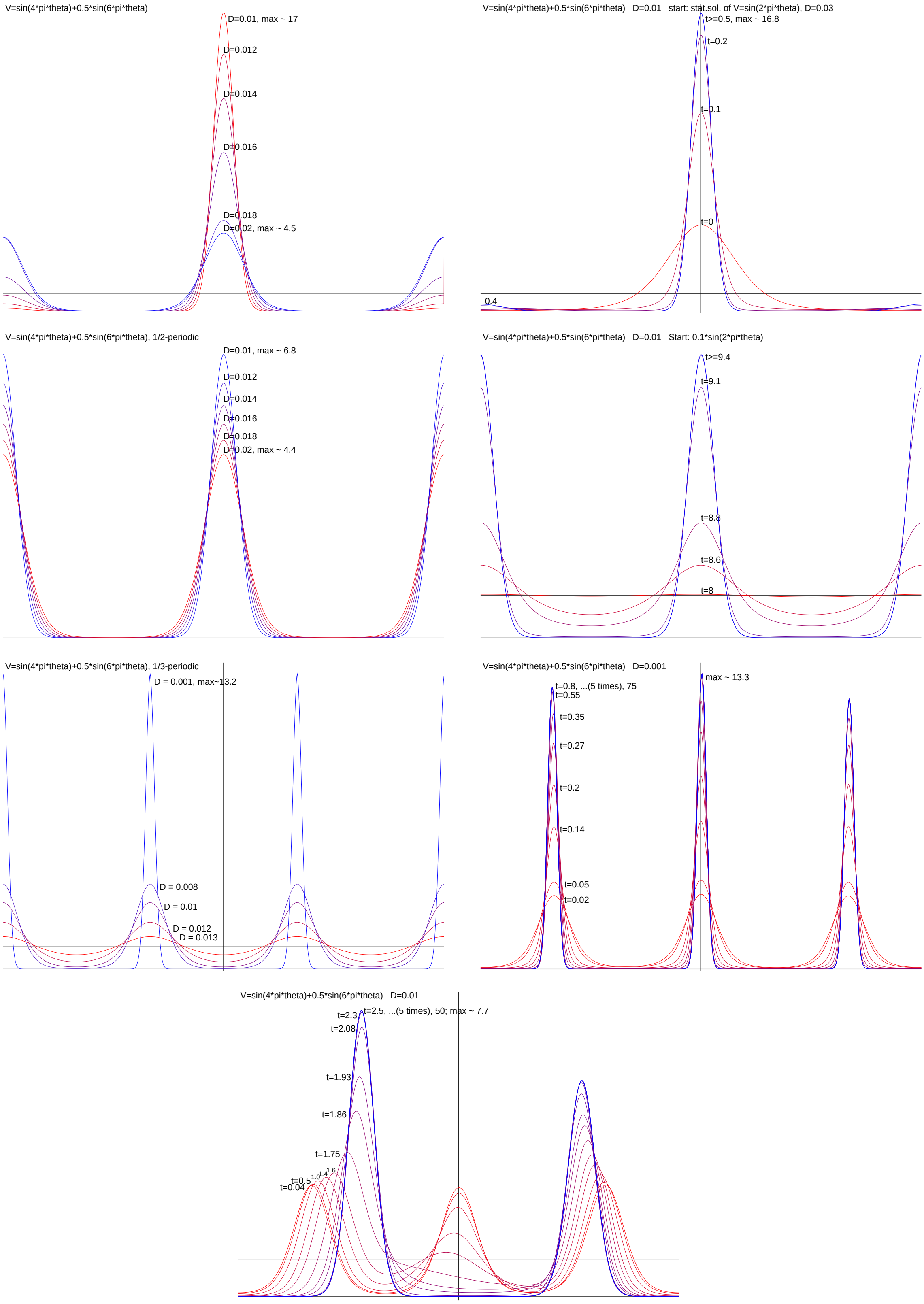

FIGURE 5. $V(\theta) = \sin(4\pi\theta) + 0.5\sin(6\pi\theta)$.

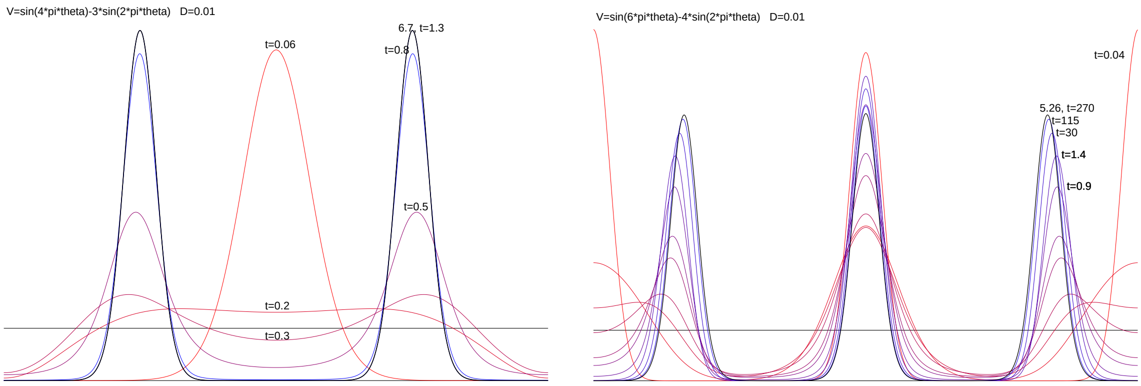

FIGURE 6. $V(\theta) = \sin(4\pi\theta) - 3\sin(2\pi\theta)$ (left), $V(\theta) = \sin(6\pi\theta) - 4\sin(2\pi\theta)$ (right) $D = 0.01$. Initial condition were one (left) and two (right) sharp peaks. Solutions were calculated with the Fourier-based algorithm. The iteration algorithm does not converge for these V's (it runs into two-cycles); however, it converges to the shown stationary solutions if $V_{(2),2}$ and $V_{(3),3}$ and $\frac{1}{2}$- and $\frac{1}{3}$-periodicity are used, respectively.

masses satisfy (11), namely $0 = \sum_{k=1}^{3} m_k V(\theta_j(t) - \theta_k(t))$ for $j = 1, 2, 3$ and $m_1 + m_2 + m_3 = 1$ ($\theta_j \in S^1$, $m_j > 0$).

7. Discussion

For the transport-diffusion equation (1) a wide variety of different patterns has been observed. Indeed, only a limited number could be shown in the last section. It emerges that it is nearly impossible to predict pattern formation only by knowing the shape of V; however, if one compiles information like the sign of the eigenvalues c_k, the zeros of V, the signs of $V'(\frac{j}{n})$ and the shapes of the V_n and of $\int_0^\theta V_n(\psi)\, d\psi$, then the picture becomes clearer.

If there is no diffusion, then we know quite something about the stability or otherwise of peak solutions. Stability of n peaks shows itself often also in the 'short-time' behavior of solutions of the diffusion-transport equation at small diffusion. Therefore it is hard to clarify numerically whether a given stationary solution is stable for small diffusion. It is an open and interesting problem how to clarify the stability of stationary solutions if diffusion is present and if several eigenvalues are positive.

A possible interest in the transport-diffusion equation (TDE) comes from its relation to the following integro-differential equation (IDE) for a function $f : [0, \infty[\times S^1 \to \mathbb{R}^+$:

$$\frac{\partial f}{\partial t}(t,\theta) = -Mf(t,\theta) + \int_{S^1}\int_{S^1} G_\sigma(\theta - \theta_o - V(\theta_i - \theta_o))\, f(t,\theta_i)\, f(t,\theta_o)\, d\theta_o\, d\theta_i, \tag{22}$$

where $M = \int_{S^1} f(0,\theta)\, d\theta$, $\sigma > 0$, $G_\sigma : S^1 \to \mathbb{R}^+$ is the periodic Gaussian with $\int_{S^1} G_\sigma(\theta)\, d\theta = 1$,

$$G_\sigma(\theta) = \frac{1}{\sqrt{2\pi}\sigma} \sum_{n\in\mathbb{Z}} e^{-\frac{1}{2}\left(\frac{\theta+n}{\sigma}\right)^2},$$

and the turning function $V : S^1 \to S^1$ is odd.

This IDE describes a jump process in which particles at an old orientation θ_o interact over S^1 with particles in θ_i and jump to a new position $\theta = \theta_o + V(\theta_i - \theta_o)$. The precision of the jump is measured by σ. Note that for the IDE V is a turning (therefore it maps to S^1), while the function V for the TDE is a velocity and takes real values that can be arbitrarily large. If, e.g., $V(\psi) = \psi$ in the IDE, then all solutions converge for $t \to \infty$ to the constant solution (Geigant [7]), while the TDE has non-constant stable stationary solutions.

Both equations preserve mass, positivity, axial symmetry and any periodicity, and both are invariant under translations and reflections. The $SO(2)$-invariance makes linearization and calculation of eigenvalues near the stationary homogeneous solution possible, as well as the fast numerical calculation of solutions by Fourier transforming the equation into a system of ODEs (see Geigant and Stoll [9] for the IDE).

Let $M = 1$. If $V = 0$, then the solutions of both systems converge to the constant 1 as $t \to \infty$ (the TDE is the linear diffusion equation, the IDE a linear jump process). If D or σ are large compared to V, solutions also converge to 1 (Theorem 3.2 for the TDE; Geigant [7] for the IDE). Therefore, if V is small, then D and σ, resp., must be very small for pattern formation. On the other hand, if $V = 0$ and $D = 0$ or $\sigma = 0$, resp., then $\frac{\partial f}{\partial t} = 0$, therefore nothing happens. Hence, if V as well as D or σ are very small, then pattern formation occurs very slowly (if at all). Last but not least, if $D = 0$ or $\sigma = 0$ but $V \neq 0$, the limiting equations of both equations have delta distributions as solutions (see Geigant [8] for the IDE).

This said, we assume that σ and V are very small, and we use Taylor expansion in σ, V to get

$$G_\sigma(\theta - \theta_o - V(\theta_i - \theta_o)) = \delta(\theta - \theta_o) - V(\theta_i - \theta_o)\delta'(\theta - \theta_o) + \frac{\sigma^2}{2}\delta''(\theta - \theta_o) + O\big((\sigma^2 + |V|)^2\big).$$

Plugging this right hand side into (22) yields the transport-diffusion equation (1) with $D = \sigma^2/2$, because

$$\begin{aligned} \int_{S^1} \delta(\theta - \theta_o) f(\theta_i) f(\theta_o)\, d\theta_i\, d\theta_o &= f(\theta), \\ \int_{S^1}\int_{S^1} \delta'(\theta - \theta_o)(V(\theta_i - \theta_o) f(\theta_o))\, d\theta_o\, f(\theta_i)\, d\theta_i &= \int_{S^1} \frac{d}{d\theta_o}\big(V(\theta_i - \theta_o) f(\theta_o)\big)\Big|_{\theta_o = \theta} f(\theta_i)\, d\theta_i \\ &= -\int_{S^1} \big(V'(\theta_i - \theta) f(\theta) - V(\theta_i - \theta) f'(\theta)\big) f(\theta_i)\, d\theta_i \\ &= -\frac{d}{d\theta}\big((V * f)(\theta)\, f(\theta)\big), \end{aligned}$$

and

$$\int_{S^1} \delta''(\theta - \theta_o) f(\theta_o) f(\theta_i)\, d\theta_o\, d\theta_i = \int_{S^1} f(\theta_i)\, d\theta_i\, f''(\theta) = f''(\theta).$$

Different arguments for this derivation are given in Mogilner and Edelstein-Keshet [14] and in Primi et al. [16].

Similarly, for the eigenvalues $\tilde{c}_k$ of (22) (see Geigant and Stoll [9]), Taylor expansion with small V yields

$$\begin{aligned}\tilde{c}_k &= -1 + 4G_{\sigma,k}\int_0^{\frac{1}{2}} \cos(\pi k\psi)\cos\big(2\pi k(\tfrac{1}{2}\psi - V(\psi))\big)\,d\psi \\ &\approx -1 + 4G_{\sigma,k}\int_0^{\frac{1}{2}} \cos(\pi k\psi)\cos(\pi k\psi)\,d\psi + 8\pi^2k^2G_{\sigma,k}\int_0^{\frac{1}{2}} \cos(\pi k\psi)\sin(\pi k\psi)V(\psi)\,d\psi \\ &= (-1+G_{\sigma,k}) + 4\pi^2k^2G_{\sigma,k}\int_0^{\frac{1}{2}} V(\psi)\sin(2\pi k\psi)\,d\psi \\ &\overset{\sigma\to 0}{\longrightarrow} 4\pi^2k^2\int_0^{\frac{1}{2}} V(\psi)\sin(2\pi k\psi)\,d\psi,\end{aligned}$$

because the k-th Fourier coefficient $G_{\sigma,k}$ tends to 1 as $\sigma \to 0$. Since $c_k = 4\pi k\int_0^{\frac{1}{2}} V(\theta)\sin(2\pi k\theta)\,d\theta$ for $D = 0$ (see (2)), the signs of the eigenvalues of both models agree for small enough V, D and σ (similar arguments were given by I. Primi, personal communication). We stress again that for larger V or D, σ the signs of the eigenvalues may differ.

But we see for example that for both models there are turning functions V that are negative on $\left]0, \frac{1}{2}\right[$ but lead to non-trivial patterns, see the Example 6.4 in Section 6.3. Especially for the IDE this was a surprise to us. Only the eigenvalue of the first mode in the IDE is always negative if V is negative, which may correspond to the result of Primi et al. [16] that there are no one-peak like solutions for small diffusion if $\int_0^{\theta} V(\psi)\,d\psi$ is negative somewhere.

The formulas for the eigenvalues show also that higher modes have larger eigenvalues for the IDE (k^2 versus k in TDE). This explains perhaps why in simulations of the IDE at small σ we see the initial formation of several peak-like maxima much more often than in simulations of the TDE with small diffusion D.

Both equations have limiting equations for $D \to 0$ and $\sigma \to 0$, respectively. For $\sigma = 0$ we have $G_\sigma = \delta_0$, and the limiting equation is

$$\frac{\partial f}{\partial t}(t,\theta) = -f(t,\theta) + \int_{S^1} f(t,\theta - V(\psi))f(t,\theta+\psi - V(\theta))\,d\psi. \tag{23}$$

In Geigant [8] it is shown that for $\sigma \to 0$ the solutions of the IDE converge to those of the limiting equation on fixed finite time intervals.

For both limiting equations a single peak is a stationary solution, which is linearly stable if V is attracting. 'Attraction' in the case of the IDE means $0 < V(\theta) < \theta$ for $0 < \theta < \frac{1}{2}$ (see Geigant [8][3]), and in the case of the TDE $V(\theta) > 0$ for $0 < \theta < \frac{1}{2}$ and $V'(0) > 0$, see Theorem 5.6. In both equations the perturbation may not extend to the opposite side of the peak since particles located there cannot turn back (because $V(\frac{1}{2}) = 0$).

Two peaks with distance $\frac{1}{2}$ whose masses add up to 1 are also a stationary solution for both limiting equations because $V(\frac{1}{2}) = 0$. For the IDE with $\sigma = 0$ Kang et al. formulate theorems on convergence of solutions to two opposite peaks if the initial distribution is sufficiently localized, see Theorems 15 and 19 in [12] and the erratum [13][4]. They show that the second moments and the supremum away from the peaks converge to zero. However, we feel that our simpler method, namely linearization near the peak, is worthwile considering.

[3]In Theorem 3.1. of [8] there is a typing mistake: 'attracting' must be defined as given here and in the definition on page 1211 in [8].

[4]Unfortunately, we still think there are assumptions missing in [13] which are needed for their proof.

For both limiting equations the assumptions for convergence to two opposite peaks are essentially an attracting shape of V near 0 ($V > 0$ to the right of 0, $V'(0) > 0$, and for the IDE additionally $V'(0) < 1$ near 0) and near $\frac{1}{2}$ ($V < 0$ to the left of $\frac{1}{2}$, $V'(\frac{1}{2}) > 0$, and for the IDE additionally $V'(\frac{1}{2}) < 1$).

It is important to see that in both limiting equations there is *no* 'mass selection' toward equal masses of the peaks. The open question is then on what time scales equalization of the peaks occurs when σ or D, resp., are positive.

The central *differences* between the two limiting equations for the TDE and IDE are as follows.

- $n \geq 2$ initial peaks, i.e., $f(0,\cdot) = \sum_{k=1}^{n} m_k \delta(. - \theta_k)$ with $m_k > 0$, do not keep that form for the IDE (e.g., starting with two peaks in θ_1, θ_2, particles jump also to positions $\theta_1 + V(\theta_2 - \theta_1)$ and $\theta_2 + V(\theta_1 - \theta_2)$).
- $n \geq 3$ peaks — even if equidistant and with equal masses — are in general *not* a stationary solution for the IDE.

Therefore, the IDE does not allow the 'peak ansatz' (see Section 5.1). Only if $V(\frac{j}{n}) = 0$ for $j = 1, \ldots, n-1$, then n equidistant peaks with *arbitrary* masses are a stationary solution of equation (23). It is an educated guess that they are locally stable *up to redistribution of mass and reorientation* if $0 < V'(\frac{j}{n}) < 1$ holds for $0 \leq j \leq n-1$.

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