Introductory Complex Analysis

Course No. 100 312

Spring 2007

Michael Stoll

Contents

Acknowledgments

These notes were written while I was teaching the *Introductory Complex Analysis* course at Jacobs University Bremen in Spring 2007. There are still some gaps near the beginning that should be filled at some point.

I would like to thank Kęstutis Česnavičius and Jessica Fintzen for making me aware of a number of errors, which are now corrected.

1. Basics

In the following, we assume that basic properties of the field of complex numbers $\mathbb{C}$ are known.

1.1. **Definition.**

(1) For $a \in \mathbb{C}$ and $r > 0$, we denote by

$$B_r(a) = \{z \in \mathbb{C} : |z - a| < r\}$$

the *open disk* of radius r around a. We write

$$\overline{B_r(a)} = \{z \in \mathbb{C} : |z - a| \leq r\}$$

for the *closed disk* of radius r around a.

(2) A subset $U \subset \mathbb{C}$ is *open*, if for every $a \in U$ there is $\varepsilon > 0$ such that $B_\varepsilon(a) \subset U$.

(3) A subset $U \subset \mathbb{C}$ is *connected*, if for every pair of open subsets $V_1, V_2 \subset \mathbb{C}$ such that $V_1 \cap V_2 = \emptyset$ and $U \subset V_1 \cup V_2$, we have $U \cap V_1 = \emptyset$ or $U \cap V_2 = \emptyset$. If U is open, this is equivalent to saying that U cannot be written as the union of two non-empty disjoint open sets.

(4) A non-empty connected open subset of $\mathbb{C}$ is called a *domain.*

Domains are the most natural domains of definition for the functions we will consider in this course.

1.2. **Definition.**

(1) Let $U \subset \mathbb{C}$. A *path* in U is a continuous map

$$\gamma : [a, b] \longrightarrow U$$

where $a < b$. The path γ is *piece-wise* $\mathcal{C}^1$, if there is a subdivision $a = a_0 < a_1 < \cdots < a_n = b$ of the interval such that γ is continuously differentiable on each subinterval $[a_k, a_{k+1}]$.

(2) We say that γ *connects* $\gamma(a)$ to $\gamma(b)$. If $\gamma(a) = \gamma(b)$, then the path is *closed.*

(3) A subset $U \subset \mathbb{C}$ is *path-connected*, if for every pair of points $w, z \in U$, there is a path in U that connects them. It is easy to see (by 'composing' paths) that this induces an equivalence relation on U.

For us, the following will be important.

1.3. **Proposition.** *A domain is path-connected.*

Proof. Let $U \subset \mathbb{C}$ be a domain. Then U is non-empty, and we can pick a point $a \in U$. Let U_1 be the subset of U of points that can be connected to a by a path in U, and let $U_2 = U \setminus U_1$. Then it is clear that U_1 and U_2 are disjoint and that their union is U. We will show that they are both open. Since U is connected, this will imply that one of U_1 and U_2 is empty, and since $a \in U_1$, it follows that $U_2 = \emptyset$, hence $U_1 = U$. So we can connect any point in U to a, and then it follows that any two points in U can be connected by a path.

It remains to show that U_1 and U_2 are both open. Let $z \in U_1$. Since U is open, there is a disk $B_\varepsilon(z) \subset U$. It is clear that any point in the disk can be connected to z, so by transitivity, $B_\varepsilon(z) \subset U_1$. This shows that U_1 is open. Now let $z \in U_2$. Again, there is a disk $B_\varepsilon(z) \subset U$. If $w \in U_1$ for some $w \in B_\varepsilon(z)$, then it would follow that $z \in U_1$ (again by transitivity), a contradiction. So $B_\varepsilon(z) \subset U_2$, and U_2 is open as well. □

2. Complex Differentiability and Holomorphic Functions

In this course, we will study functions of a complex variable that are complex differentiable. It will turn out soon that this property is *much* stronger than its real counterpart.

2.1. **Definition.** Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ a function.

(1) f is *complex differentiable* at $a \in U$, if the limit
$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$
exists (in $\mathbb{C}$). In this case, $f'(a)$ is called the *derivative* of f at a.

(2) f is *holomorphic* (on U) if f is complex differentiable at every $a \in U$. Then f' is a function $U \to \mathbb{C}$. We also write $\frac{d}{dz}f$ for f'.

(3) f is *holomorphic at (or near)* $a \in U$ if f is holomorphic on some open disk around a.

(4) A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is an *entire* function.

2.2. **Remark.** If f is holomorphic at a, then f is continuous at a (same proof as for real functions). We have the usual alternative formulation: f is complex differentiable at a with derivative $f'(a)$, if we can write
$$f(z) = f(a) + f'(a)(z - a) + (z - a)r(z)$$
for z close to a, where $\lim_{z \to a} r(z) = 0$.

We have the usual properties.

2.3. **Proposition.**

(1) *Let $f, g : U \to \mathbb{C}$ be complex differentiable at $a \in U$. Then $f + g$, $f - g$, fg are complex differentiable at a, and we have $(f \pm g)'(a) = f'(a) \pm g'(a)$ and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$. If $g(a) \neq 0$, then f/g is complex differentiable at a, and $(f/g)'(a) = (f'(a)g(a) - f(a)g'(a))/g(a)^2$.*

(2) *Let $f : U \to C$, $g : V \to \mathbb{C}$ and $a \in U$ such that $b = f(a) \in V$. If f is complex differentiable at a and g is complex differentiable at b, then $g \circ f$ is complex differentiable at a, and $(g \circ f)'(a) = g'(f(a))f'(a)$.*

(3) *Any constant function is holomorphic, with derivative zero.*
(4) *The identity function* $\mathrm{id}_{\mathbb{C}} : z \mapsto z$ *is holomorphic, with* $\mathrm{id}'_{\mathbb{C}}(z) = 1$.

Proof. The proofs are identical to those in the real setting. □

2.4. **Corollary.** *All polynomials* $p \in \mathbb{C}[z]$ *give rise to entire functions. Quotients of polynomials (rational functions) are holomorphic on* $\mathbb{C}$ *with the zeros of the denominator removed.*

Proof. This is clear from Prop. 2.3. □

3. Power Series and the Abel Limit Theorem

To get more interesting examples, we need to go beyond the algebraic operations and use analysis, i.e., sequences or series. The most important class of examples is given by power series.

3.1. **Theorem.** *Let* $(a_n)_{n\ge 0}$ *be a sequence of complex numbers. Consider the power series* $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

(1) *There is* $0 \le \rho \le \infty$ *(the radius of convergence) such that the series converges for* $|z| < \rho$ *and diverges for* $|z| > \rho$. *If* $0 < r < \rho$, *then the series converges absolutely and uniformly on* $\overline{B_r(0)}$.
(2) $\rho = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$. *If the limit exists, we have* $\rho = \lim_{n\to\infty} \dfrac{|a_n|}{|a_{n+1}|}$.
(3) *On* $B_\rho(0)$, *the function* f *defined by the series is holomorphic, and* $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. *This series has again radius of convergence* ρ.

Proof.

(1) Assume that the series converges for some $z_0 \in \mathbb{C}$, $z_0 \neq 0$. Let $r = |z_0|$. Then $|a_n| r^n$ is bounded. If $0 < r' < r$, then for $|z| \le r'$, the series converges absolutely and uniformly, since it is dominated by a constant times the geometric series in r'/r. If we set
$$\rho = \sup\{r : \text{the series converges for some } z \text{ with } |z| = r\},$$
the claim follows.
(2) Both statements follow from a comparison with a geometric series.
(3) We need to show that
$$\lim_{z\to a}\Bigl(\frac{\sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n a^n}{z-a} - \sum_{n=1}^{\infty} n a_n a^{n-1}\Bigr) = 0\,.$$
First note that
$$\frac{\sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n a^n}{z-a} = \sum_{n=1}^{\infty} a_n (z^{n-1} + a z^{n-2} + \dots + a^{n-1})\,,$$
and
$$\sum_{n=1}^{\infty} n a_n a^{n-1} = \sum_{n=1}^{\infty} a_n (a^{n-1} + a^{n-1} + \dots + a^{n-1})\,,$$

so the term under the limit is

$$\sum_{n=1}^{\infty} a_n\big((z^{n-1}-a^{n-1})+a(z^{n-2}-a^{n-2})+\cdots+a^{n-2}(z-a)\big) = (z-a)g(z,a)\,,$$

where

$$g(z,a)=\sum_{n=2}^{\infty} a_n\big(z^{n-2}+2az^{n-3}+\cdots+(n-2)a^{n-3}z+(n-1)a^{n-2}\big)\,.$$

We can assume that $|a|, |z| \le \rho' < \rho$, then $|g(z,a)| \le \sum_{n=2}^{\infty} \frac{n(n-1)}{2}|a_n|(\rho')^{n-2}$ is bounded. Therefore $(z-a)g(z,a)$ tends to zero as z tends to a, and the claim is proved. The statement about the radius of convergence of the differentiated series follows from $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

□

In the same way, we can consider power series *centered at* a of the form $\sum_{n=0}^{\infty} a_n(z-a)^n$. Such a series will converge on an open disk around a.

3.2. **Corollary.** *A function that is given on a disk $B_r(a)$ by a converging power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ has complex derivatives of every order. The coefficients are given by*

$$a_n = \frac{f^{(n)}(a)}{n!}\,.$$

Proof. This is clear from Thm. 3.1 and an easy induction. □

3.3. **Remark.** A function that is given locally (i.e., on a small disk around each point in its domain of definition) by a convergent power series is said to be *analytic.* (A similar definition can be made for real functions in several variables.) We have seen that an analytic function is holomorphic. It will turn out that the converse is true as well! Note that this is far from true for real differentiable functions.

3.4. **Abel Limit Theorem.** *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series such that $\sum_{n=0}^{\infty} a_n = a$ converges (so the radius of convergence is at least 1). Then for any $K \ge 1$, $f(z)$ tends to a as z tends to 1 within*

$$D_K = \{z \in \mathbb{C} : |z| < 1 \text{ and } |1-z| \le K(1-|z|)\}\,.$$

Proof. By replacing $f(z)$ with $f(z)-a$, we can assume that $a = 0$. Let $A_n = a_0 + a_1 + \cdots + a_n$, then $A_n \to 0$ by assumption. We use 'Abel partial summation' (setting $A_{-1} = 0$) to obtain

$$\begin{aligned}\sum_{n=0}^{N} a_n z^n = \sum_{n=0}^{N}(A_n - A_{n-1})z^n &= \sum_{n=0}^{N} A_n z^n - \sum_{n=0}^{N-1} A_n z^{n+1} \\ &= (1-z)\sum_{n=0}^{N-1} A_n z^n + A_N z^N\,.\end{aligned}$$

Note that for $|z| < 1$, the series $\sum_{n=0}^{\infty} A_n z^n$ converges (since (A_n) is bounded as a convergent sequence), that $A_N z^N \to 0$ as $N \to \infty$ and that $\sum_{n=N+1}^{\infty} a_n z^n \to 0$ as $N \to \infty$. Combining these, we get, again for $|z| < 1$,

$$f(z) = (1 - z) \sum_{n=0}^{\infty} A_n z^n \,,$$

and we have to show that this tends to zero as z approaches 1 within D_K.

Let $\varepsilon > 0$, then there is $N \in \mathbb{N}$ such that $|A_n| < \varepsilon$ for $n \geq N$. We then have

$$f(z) = (1 - z) \sum_{n=0}^{N-1} A_n z^n + (1 - z) \sum_{n=N}^{\infty} A_n z^n \,.$$

The first term tends to zero as $z \to 1$. For the second term, we have

$$\Big|(1 - z) \sum_{n=N}^{\infty} A_n z^n\Big| \leq |1 - z| \sum_{n=N}^{\infty} |A_n||z|^n \leq \varepsilon |z|^N \frac{|1 - z|}{1 - |z|} \leq \varepsilon K \,.$$

(The last inequality uses $z \in D_K$.) Since ε can be made arbitrarily small, this shows that $f(z) \to 0$. □

There is an obvious more general version where we assume that $\sum_{n=0}^{\infty} a_n z_0^n$ converges for some arbitrary $z_0 \neq 0$.

3.5. **Remark.** The 'Abel partial summation' trick can also be used to prove the following statement (Exercise).

Let (a_n) be a sequence of complex numbers such that its partial sums

$$A_n = a_0 + a_1 + \cdots + a_n$$

are bounded, and let (b_n) be a decreasing sequence of positive real numbers tending to zero. Then $\sum_{n=0}^{\infty} a_n b_n$ converges.

This has as a special case the familiar Leibniz convergence criterion for alternating series: we take $a_n = (-1)^n$.

3.6. **Examples.** Taking $f(z) = \log(1 + z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - + \ldots$, we obtain

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots = \log 2 \,.$$

Similarly, with $f(z) = \arctan z = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - + \ldots$, we get

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4} \,.$$

(Note that these series converge by Leibniz.)

Beautiful as they are, these series converge far too slowly to be of any practical use.

3.7. **Remark.** The condition "z approaches 1 within D_K", which means that we approach 1 in such a way that the angle between the line segment from 1 to z and the real axis stays bounded by a fixed acute angle, is necessary. One example is given by $f(z) = \exp(1/(z-1))$. By Thm. 5.17 below, $f(z)$ is given on $B_1(0)$ by a converging power series. It is not hard to see that $\lim_{x\uparrow 1} f(x) = 0$, but $\lim_{\varepsilon\downarrow 0} f(1-\varepsilon^2+i\varepsilon) \neq 0$ (in fact, $\lim_{\varepsilon\downarrow 0} |f(1-\varepsilon^2+i\varepsilon)| = e^{-1}$, and the previous limit does not exist). The hard part is to show that the power series converges at 1. This is a challenge problem for you!

Add: Examples (exponential function, sine, cosine, logarithm).

4. The Cauchy-Riemann Differential Equations

To be added.

5. The Cauchy Integral Theorem and its Consequences

In order to formulate our first important theorem about holomorphic functions, we need to introduce *line integrals.*

5.1. **Definition.** Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ a continuous function. Let $\gamma : [a,b] \to U$ be a path that is piece-wise $\mathcal{C}^1$. Then we define the *line integral* of f along γ by

$$\int_\gamma f(z)\,dz = \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,dt\,.$$

(The integral on the right really is a sum of integrals, one for each subinterval on which γ is continuously differentiable.)

5.2. **Examples.** Let $\gamma : [0,1] \to \mathbb{C}$, $t \mapsto e^{2\pi i t}$. Then γ is a closed path that goes around the unit circle once in the counter-clockwise direction. We find

$$\int_\gamma dz = \int_0^1 2\pi i\, e^{2\pi i t}\,dt = 0$$

(and more generally, $\int_\gamma z^n\,dz = 0$ for $n \neq -1$), but

$$\int_\gamma \frac{1}{z}\,dz = \int_0^1 \frac{1}{e^{2\pi i t}}\, 2\pi i\, e^{2\pi i t}\,dt = 2\pi i \int_0^1 dt = 2\pi i\,.$$

Here are some basic properties of line integrals.

5.3. **Lemma.** *Let* $\gamma : [a,b] \to U$ *be a path in the domain* U*,* $f : U \to \mathbb{C}$ *continuous.*

(1) *If* $\tau : [c,d] \to [a,b]$ *is monotonically increasing and piece-wise* $\mathcal{C}^1$ *invertible, then*

$$\int\limits_{\gamma\circ\tau} f(z)\,dz = \int\limits_{\gamma} f(z)\,dz\,.$$

This means that the integral is invariant under reparametrization of the path: it only depends on the image of γ *and its orientation.*

(2) *Define* $-\gamma : [-b,-a] \ni t \mapsto \gamma(-t) \in U$*. Then*

$$\int\limits_{-\gamma} f(z)\,dz = -\int\limits_{\gamma} f(z)\,dz\,.$$

(3) *If* $F : U \to \mathbb{C}$ *is holomorphic and* $F' = f$ *(so* F *is an* antiderivative *or* primitive *of* f*), then*

$$\int\limits_{\gamma} f(z)\,dz = F\big(\gamma(b)\big) - F\big(\gamma(a)\big)\,.$$

(4) *If* $|f(z)| \le M$ *on the image of* γ*, then*

$$\Big|\int\limits_{\gamma} f(z)\,dz\Big| \le M\ell(\gamma)\,,$$

where $\ell(\gamma) = \int_a^b |\gamma'(t)|\,dt$ *is the length of* γ*. This is called the* standard estimate *for the line integral.*

Proof.

(1) This follows from the change-of-variables formula for integrals.
(2) This, too.
(3) This follows from the fundamental theorem of calculus:

$$\int\limits_{\gamma} f(z)\,dz = \int\limits_a^b F'\big(\gamma(t)\big)\gamma'(t)\,dt = \int\limits_a^b \frac{d}{dt}F\big(\gamma(t)\big)\,dt = F\big(\gamma(b)\big) - F\big(\gamma(a)\big)\,.$$

(4) We have

$$\Big|\int\limits_{\gamma} f(z)\,dz\Big| = \Big|\int\limits_a^b f\big(\gamma(t)\big)\gamma'(t)\,dt\Big| \le \int\limits_a^b \big|f\big(\gamma(t)\big)\big|\,|\gamma'(t)|\,dt \le M\int\limits_a^b |\gamma'(t)|\,dt\,.$$

□

5.4. **Corollary.** *Let* f *be holomorphic on the domain* U*, such that* $f' = 0$*. Then* f *is constant.*

Proof. Let $a \in U$. Then by Prop. 1.3, for every $z \in U$, there is a path γ connecting a to z. We then have by Lemma 5.3 that

$$f(z) - f(a) = \int\limits_{\gamma} f'(z)\,dz = 0\,,$$

so $f(z) = f(a)$ is constant. □

We can now state and prove a first crucial result.

5.5. **Theorem (Cauchy Integral Theorem for Triangles).** *Let $U \subset \mathbb{C}$ be a domain, $f : U \to \mathbb{C}$ holomorphic. Let $\Delta \subset U$ be a closed triangle (i.e., the convex hull of three points), and denote by $\partial\Delta$ the closed path that goes once around the triangle in counter-clockwise orientation. Then*

$$\int_{\partial\Delta} f(z)\,dz = 0\,.$$

Proof. The idea of the proof is to construct a sequence of nested triangles, each half the size of the preceding one, and then use complex differentiability at their limit point.

So let $\Delta_0 = \Delta$, and denote by d the diameter of Δ. We proceed recursively. Suppose Δ_n has been constructed. Use the line segments connecting the midpoints of the three edges to subdivide Δ_n into four triangles $\Delta_n^{(1)}$, $\Delta_n^{(2)}$, $\Delta_n^{(3)}$ and $\Delta_n^{(4)}$, each of half the size of Δ_n. We have

$$\Big|\int_{\partial\Delta_n} f(z)\,dz\Big| = \Big|\sum_{j=1}^{4}\int_{\partial\Delta_n^{(j)}} f(z)\,dz\Big| \le 4\max_{1\le j\le 4}\Big|\int_{\partial\Delta_n^{(j)}} f(z)\,dz\Big|\,.$$

(Note that the contributions of the paths inside Δ_n cancel.) We let $\Delta_{n+1} = \Delta_n^{(j)}$ where the maximum above is attained at j. Then by induction, we have for all $n \ge 0$:

$$\Big|\int_{\partial\Delta} f(z)\,dz\Big| \le 4^n\Big|\int_{\partial\Delta_n} f(z)\,dz\Big|$$
$$\operatorname{diam}(\Delta_n) = 2^{-n}d$$
$$\ell(\partial\Delta_n) = 2^{-n}\ell(\Delta)$$

Since Δ is compact and the diameters of the triangles tend to zero, there is $a \in \Delta$ such that for every $\delta > 0$, the disk $B_\delta(a)$ contains all triangles Δ_n for n sufficiently large. Now pick $\varepsilon > 0$. Since f is complex differentiable at a, there is $\delta > 0$ such that $B_\delta(a) \subset U$ and

$$f(z) = f(a) + f'(a)(z-a) + (z-a)r(z)$$

with $|r(z)| < \varepsilon$ for all $z \in B_\delta(a)$. Note that $f(a) + f'(a)(z-a)$ has an obvious antiderivative, hence $\int_{\partial\Delta_n}\big(f(a) + f'(a)(z-a)\big)\,dz = 0$ for all n. Picking n so large that $\Delta_n \subset B_\delta(a)$, we find that

$$\Big|\int_{\partial\Delta} f(z)\,dz\Big| \le 4^n\Big|\int_{\partial\Delta_n} f(z)\,dz\Big| = 4^n\Big|\int_{\partial\Delta_n} (z-a)r(z)\,dz\Big|$$
$$\le 4^n \operatorname{diam}(\Delta_n)\ell(\partial\Delta_n)\varepsilon = d\ell(\Delta)\varepsilon\,.$$

Since $\varepsilon > 0$ was arbitrary, the claim follows. □

5.6. **Remark.** There is a converse to this statement, known as *Morera's Theorem:*

If $f : U \to \mathbb{C}$ is continuous, and $\int_{\partial\Delta} f(z)\,dz = 0$ for all closed triangles $\Delta \subset U$, then f is holomorphic.

We leave the proof as an exercise, to be attempted after reading the remainder of this section.

5.7. **Definition.** A set $U \subset \mathbb{C}$ is *star-shaped* (with respect to $a \in U$), if for every $z \in U$, the line segment connecting a to z is contained in U.

5.8. **Examples.** Any convex subset of $\mathbb{C}$ is star-shaped with respect to any of its points. The *slit plane* $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is star-shaped with respect to any point on the positive real axis. The *punctured plane* $\mathbb{C} \setminus \{0\}$ is not star-shaped with respect to any point.

5.9. **Corollary.** *Let $U \subset \mathbb{C}$ be a star-shaped domain and $f : U \to \mathbb{C}$ holomorphic. Then f has an antiderivative on U.*

Proof. Let $a \in U$ such that U is star-shaped with respect to a. We denote by $\int_a^b f(z)\,dz$ the line integral with respect to the oriented line segment from a to b. For $z \in U$, define

$$F(z) = \int_a^z f(\zeta)\,d\zeta\,;$$

since U is star-shaped, this makes sense. Now let $b \in U$, and pick $\varepsilon > 0$. Since f is continuous, there is $\delta > 0$ such that $B_\delta(b) \subset U$ and $|f(z) - f(b)| < \varepsilon$ for $z \in B_\delta(b)$. For $z \in B_\delta(b) \setminus \{b\}$, we then have by the Cauchy Integral Theorem 5.5 that

$$\Big|\frac{F(z) - F(b)}{z - b} - f(b)\Big| = \Big|\frac{1}{z - b}\int_b^z \big(f(z) - f(b)\big)\,dz\Big| \leq \frac{1}{|z - b|}|z - b|\varepsilon = \varepsilon\,.$$

This shows that F is complex differentiable at b, and $F'(b) = f(b)$. □

5.10. **Example.** The function $f(z) = 1/z$ has an antiderivative on the slit plane $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. The antiderivative that takes the value 0 at $z = 1$ is the *principal branch of the logarithm.*

On the other hand, there is no antiderivative on $\mathbb{C} \setminus \{0\}$: otherwise, we would have

$$\int_{\partial B_1(0)} \frac{dz}{z} = 0\,,$$

but the integral evaluates to $2\pi i$.

5.11. **Theorem (Cauchy Integral Theorem for Star-Shaped Domains).** *Let $U \subset \mathbb{C}$ be a star-shaped domain and $f : U \to \mathbb{C}$ holomorphic. Let $\gamma : [a, b] \to U$ be a closed path. Then*

$$\int_\gamma f(z)\,dz = 0\,.$$

Proof. By Cor. 5.9, f has an antiderivative F on U. Since γ is closed, by Lemma 5.3, we get

$$\int_\gamma f(z)\,dz = F\big(\gamma(b)\big) - F\big(\gamma(a)\big) = 0\,.$$

□

5.12. **Theorem (Cauchy Integral Formula for Disks).** *Let U be a domain, $f : U \to \mathbb{C}$ holomorphic. Let $a \in U$ and $r > 0$ such that $\overline{B_r(a)} \subset U$. Then for all $z \in B_r(a)$, we have*

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial B_r(a)} \frac{f(\zeta)}{\zeta - z}\, d\zeta\,.$$

(As usual, the path of integration goes around the circle counter-clockwise.)

Proof. Let $z \in B_r(a)$. We pick $\varepsilon > 0$ so small that $\overline{B_\varepsilon(z)} \subset B_r(a)$. We connect the two circles by two line segments contained in the line ℓ through a and z. We obtain two domains whose union is $B_r(a) \setminus (\ell \cup \overline{B_\varepsilon(z)})$. Let γ_1 and γ_2 denote their boundaries, oriented counter-clockwise. Then

$$\int\limits_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int\limits_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int\limits_{\partial B_r(a)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int\limits_{\partial B_\varepsilon(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta\,.$$

Each of the paths γ_1 and γ_2 is contained in a star-shaped domain on which f is holomorphic. (There is a larger open disk $\overline{B_r(a)} \subset B_{r'}(a) \subset U$; remove any line segment joining z to its boundary, not contained in ℓ and not meeting γ_j.) So by Thm. 5.11, the integrals along γ_1 and γ_2 vanish, and we get

$$\int\limits_{\partial B_r(a)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int\limits_{\partial B_\varepsilon(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int\limits_{\partial B_\varepsilon(z)} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta + f(z) \int\limits_{\partial B_\varepsilon(z)} \frac{1}{\zeta - z}\, d\zeta\,.$$

The second integral easily evaluates to $2\pi i$, compare the example in 5.2. Near z, the function under the first integral is bounded, hence the integral is bounded by a constant times ε. Since we can make ε as small as we like, the claim follows. □

Before we state the next consequence, we need to remind ourselves of a result connecting integrals and sequences of functions.

5.13. **Lemma.** *Let $f_n : U \to \mathbb{C}$ be a sequence of continuous functions on a domain U. Let $\gamma : [a, b] \to U$ be a path and assume that f_n converges uniformly to a function f on the image of γ. Then*

$$\lim_{n\to\infty} \int\limits_{\gamma} f_n(z)\, dz = \int\limits_{\gamma} f(z)\, dz\,.$$

Proof. Let $\varepsilon > 0$. Then there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|f_n(\gamma(t)) - f(\gamma(t))| < \varepsilon$ for all $n \geq N$ and $t \in [a, b]$. By the standard estimate, we then find

$$\Big|\int\limits_{\gamma} f_n(z)\, dz - \int\limits_{\gamma} f(z)\, dz\Big| \leq \int\limits_{\gamma} |f_n(z) - f(z)|\, dz \leq \varepsilon \ell(\gamma)\,,$$

which implies the claim. □

5.14. **Proposition.** *Under the assumptions of Thm. 5.12, we have that*

$$f'(z) = \frac{1}{2\pi i} \int_{\partial B_r(a)} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta$$

for all $z \in B_r(a)$.

Proof. Let $w, z \in B_r(a)$, $w \neq z$. Then by the Cauchy Integral Formula 5.12, we have

$$\frac{f(w) - f(z)}{w - z} = \frac{1}{2\pi i} \int_{\partial B_r(a)} \frac{f(\zeta)}{(\zeta - w)(\zeta - z)} \, d\zeta \, .$$

Pick a sequence $w_n \to z$ in $B_r(a)$. If $w_n, z \in \overline{B_\rho(a)}$ with $0 < \rho < r$ and $|f(\zeta)| \leq M$ on $\partial B_r(a)$, then

$$\left| \frac{f(\zeta)}{(\zeta - w_n)(\zeta - z)} - \frac{f(\zeta)}{(\zeta - z)^2} \right| = \left| \frac{f(\zeta)(z - w_n)}{(\zeta - w_n)(\zeta - z)^2} \right| \leq \frac{M}{(r - \rho)^3} \, |z - w_n| \, ,$$

hence $f(\zeta)/((\zeta - w_n)(\zeta - z))$ converges uniformly as a function of ζ on $\partial B_r(a)$ to $f(\zeta)/(\zeta - z)^2$. By Lemma 5.13, we then have

$$f'(z) = \lim_{n \to \infty} \frac{f(w_n) - f(z)}{w_n - z} = \frac{1}{2\pi i} \int_{\partial B_r(a)} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta \, .$$

□

5.15. **Corollary.** *Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ holomorphic. Then f' is again holomorphic on U. In particular, f has complex derivatives of any order on U.*

Proof. Let $a \in U$; then there is $r > 0$ such that $\overline{B_r(a)} \subset U$. By Prop. 5.14, for $z \in B_r(a)$, we have

$$f'(z) = \frac{1}{2\pi i} \int_{\partial B_r(a)} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta \, .$$

We show that f' is complex differentiable at a: for z close to a, we have

$$\frac{f'(z) - f'(a)}{z - a} = \frac{1}{2\pi i} \int_{\partial B_r(a)} \left(\frac{f(\zeta)}{(\zeta - a)(\zeta - z)^2} + \frac{f(\zeta)}{(\zeta - a)^2(\zeta - z)} \right) d\zeta \, .$$

As z tends to a, the integrand converges uniformly on $\partial B_r(a)$, hence the complex derivative of f' at a exists. The second claim then follows by induction. □

5.16. **Corollary.** *Under the assumptions of Thm. 5.12, we have for $n \geq 0$ and $z \in B_r(a)$ that*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(a)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta \, .$$

Proof. This proceeds by induction. Using the arguments in the proofs of Cor. 5.14 and Cor. 5.15, we find that

$$f^{(n)}(z) = \frac{(n-1)!}{2\pi i} \frac{d}{dz} \int\limits_{\partial B_r(a)} \frac{f(\zeta)}{(\zeta - z)^n}\, d\zeta$$
$$= \frac{(n-1)!}{2\pi i} \int\limits_{\partial B_r(a)} \frac{d}{dz} \frac{f(\zeta)}{(\zeta - z)^n}\, d\zeta = \frac{n!}{2\pi i} \int\limits_{\partial B_r(a)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta\,.$$

□

We see that a holomorphic function is even $\mathcal{C}^\infty$. But even more is true.

5.17. **Theorem.** *Let $U \subset \mathbb{C}$ be a domain, $f : U \to \mathbb{C}$ holomorphic. Assume that $\overline{B_r(a)} \subset U$. Then on $B_r(a)$, f is given by a power series:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

with radius of convergence at least r. We have $a_n = f^{(n)}(a)/n!$.

Proof. After a translation, we can assume (for simplicity) that $a = 0$. We expand the integrand in the Cauchy Integral Formula 5.12 as a power series in z. Let $z \in B_r(0)$; then there is $0 < \rho < r$ such that $|z| < \rho$. We then have

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial B_\rho(0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int\limits_{\partial B_\rho(0)} \frac{f(\zeta)}{\zeta} \frac{1}{1 - z/\zeta}\, d\zeta$$
$$= \frac{1}{2\pi i} \int\limits_{\partial B_\rho(0)} \frac{f(\zeta)}{\zeta} \sum_{n=0}^{\infty} \frac{z^n}{\zeta^n}\, d\zeta = \frac{1}{2\pi i} \int\limits_{\partial B_\rho(0)} \sum_{n=0}^{\infty} \Big(\frac{f(\zeta)}{\zeta^{n+1}}\Big) z^n\, d\zeta\,.$$

If $|z| \le \rho' < \rho$ and $|f(\zeta)| \le M$ on $\partial B_\rho(0)$, then the nth term of the series under the integral is bounded by $M/\rho(\rho'/\rho)^n$, hence by the Weierstrass M-test, the series converges uniformly. By Lemma 5.13, we can therefore interchange integral and summation to obtain, using Cor. 5.16,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \Big(\int\limits_{\partial B_\rho(0)} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta \Big) z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n\,.$$

Since the series converges on $B_r(0)$, its radius of convergence is at least r. □

This says that *a holomorphic function is analytic.*

5.18. **Corollary (Cauchy's Bound on Taylor Coefficients).** *If $f : U \to \mathbb{C}$ is holomorphic and $\overline{B_r(a)} \subset U$, then for the coefficients a_n in the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$, we have the bound*

$$|a_n| \le \frac{M}{r^n}\,,$$

where $|f(\zeta)| \le M$ for $|\zeta - a| = r$.

Proof. This follows from the formula in Cor. 5.16 by the standard estimate:

$$|a_n| = \frac{|f^{(n)}(a)|}{n!} = \Big|\frac{1}{2\pi i}\int\limits_{\partial B_r(a)} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta\Big| \le \frac{1}{2\pi}\frac{M}{r^{n+1}}\cdot 2\pi r = \frac{M}{r^n}\,.$$

□

5.19. **Corollary (Liouville's Theorem).** *A bounded entire function is constant.*

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and bounded: $|f(z)| \le M$ for all $z \in \mathbb{C}$. By Thm. 5.17, f is given by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

that converges everywhere. Then for every $r > 0$, we have by Cor. 5.18 that

$$|a_n| \le \frac{M}{r^n}\,.$$

Letting $r \to \infty$, we see that $a_n = 0$ for $n \ge 1$, hence $f(z) = a_0$ is constant. □

5.20. **Corollary (Fundamental Theorem of Algebra).** *Let $p \in \mathbb{C}[z]$ be a non-constant polynomial. Then p has a root in $\mathbb{C}$.*

Proof. Let $p \in \mathbb{C}[z]$ and assume p does not have a root in $\mathbb{C}$. Then $f(z) = 1/p(z)$ is an entire function. Write

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

with $a_n \neq 0$. Then

$$p(z) = a_n z^n\Big(1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\Big)\,,$$

so there is $R > 0$ such that for $|z| \ge R$ we have $|p(z)| \ge |a_n|R^n/2 \ge |a_n|/2$. Hence $|f(z)| \le 2/|a_n|$ is bounded for $|z| \ge R$. On the other hand, $\overline{B_R(0)}$ is compact, hence $f(z)$ is also bounded for $|z| \le R$. So f is a bounded entire function and therefore constant by Liouville's Theorem 5.19. Hence $p(z) = 1/f(z)$ is constant as well. □

6. Local Behavior of Holomorphic Functions

We will study the behavior of holomorphic functions near their zeros. But first we need a result about the behavior at the other points.

6.1. **Lemma.** *Let g be holomorphic near $a \in \mathbb{C}$ such that $g(a) \neq 0$. Let $k \ge 1$. Then there is $\varepsilon > 0$ and a holomorphic function $h : B_\varepsilon(a) \to \mathbb{C}$ such that $g = h^k$ on $B_\varepsilon(a)$.*

Proof. Write $g(z) = \alpha g_1(z)$ with $g_1(a) = 1$ (so $\alpha = g(a)$). Since g_1 is continuous, there is $\varepsilon > 0$ such that g_1 is defined on $B_\varepsilon(a)$ and $|g_1(z) - 1| < 1$ for $z \in B_\varepsilon(a)$. On $B_1(1)$, the principal branch of the logarithm is defined (by the usual power series). Picking a kth root β of α, we set on $B_\varepsilon(a)$

$$h(z) = \beta \exp\Big(\frac{\log g_1(z)}{k}\Big)\,;$$

then $h(z)^k = \beta^k \exp(\log g_1(z)) = \alpha g_1(z) = g(z)$. □

6.2. **Definition.** Let f be holomorphic near $a \in \mathbb{C}$. Then we call

$$\operatorname{ord}_a f = \min\{n \in \mathbb{N} : f^{(n)}(a) \neq 0\}$$

the *order of vanishing* of f at a.

Note that $\operatorname{ord}_a f > 0$ if and only if $f(a) = 0$. When $\operatorname{ord}_a f = 1$, we say that f has a *simple zero* at a.

6.3. **Lemma.** *Let f be holomorphic near a. If $\operatorname{ord}_a f = \infty$, then $f = 0$ on a neighborhood of a.*

Proof. On some disk around a, f is holomorphic. Therefore, f is given there by the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n .$$

If $\operatorname{ord}_a f = \infty$, then all coefficients vanish, hence $f = 0$ on that disk. □

6.4. **Lemma.** *Let f be holomorphic near $a \in \mathbb{C}$, with a simple zero at a. Then for every sufficiently small $\varepsilon > 0$, there is an open neighborhood V_ε of a such that $f : V_\varepsilon \to B_\varepsilon(0)$ is biholomorphic.*

Proof. Since $f'(a) \neq 0$, f has an inverse near a, which is again holomorphic (e.g., first construct it as a real differentiable function, then note that the derivative is the inverse of multiplication by a complex number (the derivative of f) and therefore complex linear). So for $\varepsilon > 0$ sufficiently small, taking $V_\varepsilon = f^{-1}(B_\varepsilon(0))$ we get the result. □

Now we consider the case that f has a zero at a, but is not zero in a whole neighborhood.

6.5. **Theorem.** *Let f be holomorphic near $a \in \mathbb{C}$ such that $0 < k = \operatorname{ord}_a f < \infty$. Then there is $\varepsilon > 0$ and a holomorphic function h on $B_\varepsilon(a)$ such that f is defined on $B_\varepsilon(a)$ and $f = h^k$ there, $h(a) = 0$, and $h'(a) \neq 0$.*

Proof. On a small disk around a, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = (z-a)^k \sum_{n=k}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^{n-k} = (z-a)^k g(z)$$

where g is holomorphic and $g(a) = f^{(k)}(a)/k! \neq 0$. By Lemma 6.1, on some $B_\varepsilon(a)$, there is a holomorphic function h_1 such that $g = h_1^k$. We have $h_1(a) \neq 0$. If we set $h(z) = (z-a)h_1(z)$, then $f(z) = h(z)^k$, $h(a) = 0$, and $h'(a) = h_1(a) \neq 0$. □

6.6. **Corollary.** *In the situation of Thm. 6.5, the following holds. For every sufficiently small $\varepsilon > 0$, there is an open neighborhood V_ε of a such that f maps V_ε surjectively onto $B_\varepsilon(0)$ and such that $f^{-1}(0) \cap V_\varepsilon = \{a\}$ and for every $0 \neq w \in B_\varepsilon(0)$, we have $\#(f^{-1}(w) \cap V_\varepsilon) = k$.*

So every sufficiently small $w \neq 0$ has exactly k preimages under f near a.

Proof. By Lemma 6.4, for every sufficiently small $\varepsilon > 0$, there is a neighborhood V_ε of a such that $h : V_\varepsilon \to B_{\sqrt[k]{\varepsilon}}(0)$ is biholomorphic. In particular, h is a bijection between these sets. Now the claim follows from $f = h^k$. □

The following is a kind of converse to Cor. 6.4.

6.7. **Corollary.** *If f is holomorphic and injective near a, then $f'(a) \neq 0$.*

Proof. Let $g(z) = f(z) - f(a)$; then $g(a) = 0$, and g is injective near a. By the preceding results, this implies $\operatorname{ord}_a g = 1$, hence $f'(a) = g'(a) \neq 0$. □

The following important result states that zeros of finite order are isolated.

6.8. **Corollary.** *If f is holomorphic near $a \in \mathbb{C}$ and $\operatorname{ord}_a f < \infty$, then there is $\varepsilon > 0$ such that $f(z) \neq 0$ for $z \in B_\varepsilon(a) \setminus \{a\}$.*

If there is a sequence $z_n \to a$, $z_n \neq a$, such that $f(z_n) = 0$ for all n, then $f = 0$ on a neighborhood of a.

Proof. If $f(a) \neq 0$, this follows from the continuity of f. Otherwise, it is a consequence of Cor. 6.6. The second statement follows from the first and Lemma 6.3. □

The following result shows (again) that holomorphic functions are very 'rigid': they are determined by a rather small amount of information.

6.9. **Theorem.** *Let $U \subset \mathbb{C}$ be a domain, $f, g : U \to \mathbb{C}$ holomorphic. Assume that*

(i) *$f|_A = g|_A$ on a subset $A \subset U$ that has an accumulation point in U, or that*
(ii) *$f^{(n)}(a) = g^{(n)}(a)$ for some $a \in U$ and all $n \geq 0$.*

Then $f = g$ on U.

Proof. We set $h = f - g$. Then we have to show that $h = 0$ on U. Let

$$U_1 = \{z \in U : h = 0 \text{ on a neighborhood of } z\} \quad \text{and} \quad U_2 = U \setminus U_1 .$$

I claim that both U_1 and U_2 are open, and that $U_1 \neq \emptyset$. Since U is a domain (open and connected), it follows that $U_2 = \emptyset$, i.e., $U_1 = U$, and therefore $h = 0$ on U.

U_1 is open essentially by definition: let $z \in U_1$, then there is $\varepsilon > 0$ such that $h = 0$ on $B_\varepsilon(z)$. But this implies $B_\varepsilon(z) \subset U_1$, since for $w \in B_\varepsilon(z)$, we have $B_\delta(w) \subset B_\varepsilon(z)$, where $\delta = \varepsilon - |w - z|$.

To show that U_2 is open, let $z \in U_2$. Then h does not vanish identically near z, so $\operatorname{ord}_z f < \infty$. Hence by Cor. 6.8, $h(w) \neq 0$ for $w \in B_\varepsilon(z) \setminus \{z\}$ if $\varepsilon > 0$ is small enough. But then for each such w, h does not even have a zero on $B_{|w-z|}(w)$, so $B_\varepsilon(z) \subset U_2$.

Finally, we show that $U_1 \neq \emptyset$. In case (ii), we have $\operatorname{ord}_a h = \infty$, so $a \in U_1$ by Lemma 6.3. In case (i), we let $a \in U$ be an accumulation point of A and use Cor. 6.8 to obtain the same conclusion. □

6.10. **Examples.** The condition that A has an accumulation point in U is necessary, as the examples $U = \mathbb{C}$, $A = \mathbb{Z}$, $f(z) = 0$, $g(z) = \sin(\pi z)$ or $U = \mathbb{C} \setminus \{0\}$, $A = \{1/n : 0 \neq n \in \mathbb{N}\}$, $f(z) = 0$, $g(z) = \sin(\pi/z)$ show.

6.11. **Remark.** In fact, a subset $A \subset U$ has the property that for holomorphic functions $f, g : U \to \mathbb{C}$, $f|_A = g|_A$ implies $f = g$ if and only if A has an accumulation point in U: we will see later that otherwise, there is a holomorphic function $f \neq 0$ on U such that $f|_A = 0$.

6.12. **Theorem.** *A non-constant holomorphic function f is open (i.e., the image under f of an open set is open).*

Proof. Let $U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}$ holomorphic and non-constant (everywhere, i.e., on a neighborhood of every point in U). Let $a \in U$, $b = f(a)$. We have $0 < \operatorname{ord}_a(f - b) < \infty$, hence by Cor. 6.6, there is a neighborhood V of a and $\varepsilon > 0$ such that $(f - b)(V) = B_\varepsilon(0)$, i.e., $f(V) = B_\varepsilon(b)$. So $B_\varepsilon(b) \subset f(U)$. Since $b \in f(U)$ was arbitrary, $f(U)$ is open. □

6.13. **Corollary (Maximum Principle).** *Let $U \subset \mathbb{C}$ be a domain, $f : U \to \mathbb{C}$ holomorphic. If f is non-constant, then $|f(z)|$ does not have a maximum in U.*

If $K \subset U$ is compact, then $|f(z)|$ attains its maximum on K on the boundary ∂K. If the maximum is also obtained in the interior, then f is constant.

Proof. Assume that $|f(z)|$ has a maximum at $a \in U$. By Thm. 6.12, then $f(U)$ contains a neighborhood of $f(a)$. But this neighborhood will contain points of larger absolute value than $f(a)$, a contradiction.

The second statement follows by considering f on the interior of K. □

6.14. **Corollary (Schwarz Lemma).** *Let $f : B_1(0) \to B_1(0)$ be holomorphic with $f(0) = 0$. Then $|f(z)| \le |z|$ for all $z \in B_1(0)$ and $|f'(0)| \le 1$. If we have $|f(z_0)| = |z_0|$ for some $0 \ne z_0 \in B_1(0)$ or $|f'(0)| = 1$, then f is a rotation: $f(z) = e^{i\phi} z$ for some $\phi \in \mathbb{R}$.*

Proof. We can write $f(z) = zg(z)$ with g holomorphic on $B_1(0)$. For $0 < |z| = r < 1$, we have $|g(z)| = |f(z)|/|z| \le 1/r$. By the maximum principle Cor. 6.13, this implies that $|g(z)| \le 1/r$ for all $|z| \le r$. Letting $r \to 1$, we see that $|g(z)| \le 1$ for all $|z| < 1$. This implies the first statement (note that $f'(0) = g(0)$).

If we have equality somewhere, then g must be constant (since then $|g(z)|$ attains its maximum on $B_1(0)$), and the constant has absolute value 1, so is of the form $e^{i\phi}$ with $\phi \in \mathbb{R}$. Hence $f(z) = zg(z) = e^{i\phi} z$. □

7. Isolated Singularities

7.1. **Definition.** Let $a \in \mathbb{C}$, $U \subset \mathbb{C}$ an open neighborhood of a. If f is holomorphic on $U \setminus \{a\}$, then a is an *isolated singularity* of f.

7.2. **Definition.** Let a be an isolated singularity of f, where f is holomorphic on $U \setminus \{a\}$ as above.

(1) a is a *removable singularity* of f, if there is a holomorphic function $\tilde{f}$ on U such that $\tilde{f} = f$ on $U \setminus \{a\}$.
(2) a is a *pole* of f if a is not a removable singularity and there is $m \ge 0$ such that $(z - a)^m f(z)$ has a removable singularity at a. The smallest such m is called the *order* of the pole.
(3) If none of the above holds, then a is an *essential singularity* of f.

Typical examples are (1) $z/(e^z - 1)$ at $a = 0$, (2) $1/z^m$ at $a = 0$ (for $m \ge 1$), (3) $\exp(1/z)$ at $a = 0$.

We first deal with removable singularities.

7.3. **Theorem.** *If a is an isolated singularity of f, and f is bounded near a, then a is a removable singularity of f.*

Proof. Let $\varepsilon > 0$ such that $\overline{B_\varepsilon(a)} \subset U$, where U is a neighborhood of a such that f is holomorphic on $U \setminus \{a\}$. By assumption, f is bounded on $\overline{B_\varepsilon(a)}$. In the same way as in the proof of the Cauchy Integral Formula 5.12, we find that for $a \neq z \in B_\varepsilon(a)$, we have

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial B_\varepsilon(a)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int\limits_{\partial B_\delta(a)} \frac{f(\zeta)}{\zeta - z}\, d\zeta\,,$$

where $0 < \delta < |z - a|$. Now let M be a bound for f on $\overline{B_\varepsilon(a)}$. Then

$$\Big|\frac{1}{2\pi i} \int\limits_{\partial B_\delta(a)} \frac{f(\zeta)}{\zeta - z}\, d\zeta\Big| \le \frac{\delta M}{|z-a| - \delta} \to 0 \quad \text{as } \delta \downarrow 0,$$

so

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial B_\varepsilon(a)} \frac{f(\zeta)}{\zeta - z}\, d\zeta\,.$$

The right hand side makes sense for $z = a$; define $\tilde{f}$ on $B_\varepsilon(a)$ by the right hand side (and $\tilde{f} = f$ elsewhere). Then $\tilde{f}$ is a holomorphic (compare Prop. 5.14) extension of f to U. ☐

Now let us consider poles.

7.4. **Theorem.** *Let a be an isolated singularity of f.*

(1) *If $|f(z)| \to \infty$ as $z \to a$, then a is a pole of f.*
(2) *If a is a pole of order m of f, then $f(z) = h(z)/(z-a)^m$ on $U \setminus \{a\}$, where h is holomorphic on U and $h(a) \neq 0$.*

Proof. Assume that $|f(z)| \to \infty$ as $z \to a$. Then on a sufficiently small punctured disk around a, $f(z) \neq 0$, hence $1/f(z)$ is defined on the punctured disk and is bounded there. By Thm. 7.3, a is a removable singularity of $1/f$, and $g(a) = 0$, where g is the extension of $1/f$ into a. so $g(z) = (z-a)^m h_1(z)$ near a for some $m > 1$ and a holomorphic function h_1 such that $h_1(a) \neq 0$. Then $f(z) = h(z)/(z-a)^m$ near a, where $h = 1/h_1$. Since we can define $h(z) = (z-a)^m f(z)$ for $z \in U \setminus \{a\}$. the representation is valid on all of $U \setminus \{a\}$. It is then clear that f has a pole of order m at a.

Conversely, if a is a pole of order m of f, then $(z-a)^m f(z)$ has a removable singularity, so $h(z) = (z-a)^m f(z)$ extends to a holomorphic function on U. We must have $h(a) \neq 0$; otherwise we can reduce m. Hence $f(z) = h(z)/(z-a)^m$ as claimed. It is then clear that $|f(z)| \to \infty$ as $z \to a$. ☐

So poles are in some sense well-behaved and constitute fairly harmless singularities.

7.5. **Definition.** Let $U \subset \mathbb{C}$ be a domain. A function f that is holomorphic on U except for isolated singularities that are poles is said to be *meromorphic* in U.

If a is a pole of f, we define $\operatorname{ord}_a f = -m$, where m is the order of the pole. Then for functions f and g that are meromorphic near a, we have $\operatorname{ord}_a(fg) = \operatorname{ord}_a f + \operatorname{ord}_a g$.

7.6. Remark.

(1) If f is meromorphic on U, then neither the zeros nor the poles accumulate at a point in U.
(2) Thm. 6.9 holds for meromorphic functions.
(3) The set of all meromorphic functions on a domain U forms a field (with respect to point-wise addition and multiplication).

Indeed, it is clear that we have a ring. It remains to show that if $f \neq 0$, then $1/f$ is again meromorphic. But zeros and poles of f are isolated in U (U is connected, so if f is locally zero somewhere, f must be zero on U) and turn into poles and zeros of $1/f$.

As to essential singularities, we have the following result, which shows that they are (in some precise sense) as bad as they can possibly be.

7.7. Theorem (Casorati-Weierstrass). *Let a be an essential singularity of f. Then for all sufficiently small $\varepsilon > 0$, $f(B_\varepsilon(a) \setminus \{a\})$ is dense in $\mathbb{C}$.*

Proof. Let $\varepsilon > 0$ be so small that f is holomorphic on $B_\varepsilon(a) \setminus \{a\}$. Assume that $f(B_\varepsilon(a) \setminus \{a\})$ is not dense in $\mathbb{C}$. Then there is $b \in \mathbb{C}$ and $\delta > 0$ such that $B_\delta(b) \cap f(B_\varepsilon(a) \setminus \{a\}) = \emptyset$. Now consider

$$g(z) = \frac{1}{f(z) - b} \qquad \text{for } z \in B_\varepsilon(a) \setminus \{a\}.$$

Since $|f(z) - b| \geq \delta$ for all relevant z, g is bounded (by $1/\delta$) on $B_\varepsilon(a) \setminus \{a\}$ and therefore has a removable singularity at a. But then

$$f(z) = \frac{1}{g(z)} + b$$

has at worst a pole at a, contradicting the assumption. □

7.8. Remark. In fact, much more is true: *Picard's Theorem* says that either $f(B_\varepsilon(a) \setminus \{a\}) = \mathbb{C}$, or $f(B_\varepsilon(a) \setminus \{a\}) = \mathbb{C} \setminus \{b\}$ for some $b \in \mathbb{C}$.

An example of the former is $a = 0$, $f(z) = \sin(1/z)$, an example of the latter is $a = 0$, $f(z) = \exp(1/z)$ (with $b = 0$).

If f has a pole at a, then $f(z) = g(z)/(z-a)^m$ with g holomorphic near a, $g(a) \neq 0$. Then g has a Taylor series near a:

$$g(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \quad c_0 \neq 0.$$

Hence, for $a \neq z$ near a, we have

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-a)^n \qquad \text{with } a_n = c_{n+m},\ a_{-m} \neq 0.$$

This is a special case of a Laurent series.

7.9. **Definition.** A *Laurent series* centered at $a \in \mathbb{C}$ is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-a)^n \,.$$

$\sum_{n=-\infty}^{-1} a_n(z-a)^n = \sum_{n=1}^{\infty} a_{-n}/(z-a)^n$ is called the *principal part* of the series. The series is said to be (absolutely, uniformly, ...) *convergent* if both

$$\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n} \quad \text{and} \quad \sum_{n=0}^{\infty} a_n(z-a)^n$$

are. In this case, the value of the series is the sum of the values of the two parts.

7.10. **Remark.** The series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges in $B_R(a)$, where

$$R = 1/\limsup_{n\to\infty} \sqrt[n]{|a_n|}\,,$$

and the series $\sum_{n=-\infty}^{-1} a_n(z-a)^n$ converges in $\mathbb{C} \setminus \overline{B_r(a)}$, where

$$r = \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}\,.$$

So if $r < R$, the Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-a)^n$ converges absolutely (and uniformly on compact subsets) on the *annulus* $A_{r,R}(a) = \{z \in \mathbb{C} : r < |z-a| < R\}$. It defines a holomorphic function f there, and we have

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n (z-a)^{n-1}\,.$$

Note that the series for f' has no $1/(z-a)$ term.

If $r \ge R$, then the Laurent series does not converge on any open set, and if $r > R$, it does not converge anywhere.

We have now an analogue for annuli of the Taylor expansion on open disks.

7.11. **Theorem.** *Let f be holomorphic on the domain $U \subset \mathbb{C}$, and assume that $A = A_{r,R}(a) \subset U$. Then f is given on A by a convergent Laurent series centered at a: we have*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n \quad \textit{with} \quad a_n = \frac{1}{2\pi i} \int_{\partial B_\rho(a)} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, d\zeta$$

for any $r < \rho < R$.

Proof. Without loss of generality, we may assume that $a = 0$. Let $z \in A$, let $\delta, \varepsilon > 0$ such that $r + \delta + \varepsilon < |z| < R - \delta - \varepsilon$. Then by a by now already familiar argument, we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_{\partial B_{R-\delta}(0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\partial B_{r+\delta}(0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta\,.$$

In the first integral on the right, we substitute $1/(\zeta - z) = \sum_{n=0}^{\infty} z^n/\zeta^{n+1}$ (valid for $|z| < |\zeta|$) and obtain

$$\frac{1}{2\pi i} \int_{\partial B_{R-\delta}(0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_{\partial B_{R-\delta}(0)} \sum_{n=0}^{\infty} \frac{f(\zeta)}{\zeta^{n+1}} z^n\, d\zeta = \sum_{n=0}^{\infty} \Bigl(\frac{1}{2\pi i} \int_{\partial B_{R-\delta}(0)} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta\Bigr) z^n\,.$$

In the second integral, we substitute $1/(\zeta - z) = -\sum_{n=1}^{\infty} \zeta^{n-1}/z^n$ (valid for $|z| > |\zeta|$) and obtain

$$-\frac{1}{2\pi i} \int\limits_{\partial B_{r+\delta}(0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int\limits_{\partial B_{r+\delta}(0)} \sum_{n=1}^{\infty} \frac{f(\zeta)}{\zeta^{-n+1}} z^{-n}\, d\zeta = \sum_{n=-\infty}^{-1} \Big(\frac{1}{2\pi i} \int\limits_{\partial B_{r+\delta}(0)} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta \Big) z^n .$$

Finally, observe that

$$\int\limits_{\partial B_\rho(0)} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta$$

is independent of $r < \rho < R$ (the difference of two such integrals can be written as a sum of closed line integrals in star-shaped subdomains of A). □

In particular, if f has an isolated singularity at a, then on some punctured disk $B_\varepsilon(a) \setminus \{a\}$, f is given by a Laurent series.

7.12. **Definition.** In this situation, the principal part of this Laurent series is called the *principal part* of the singularity.

7.13. **Remark.** Let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ on $B_\varepsilon(a) \setminus \{a\}$. Then

$$\begin{aligned}
a \text{ is removable} &\iff a_n = 0 \text{ for all } n < 0 \\
&\iff \text{the principal part is zero;} \\
a \text{ is a pole} &\iff a \text{ is not removable and } a_n = 0 \text{ for } n \ll 0 \\
&\iff \text{the principal part is a non-zero finite sum;} \\
a \text{ is essential} &\iff a_n \neq 0 \text{ for infinitely many } n < 0 \\
&\iff \text{the principal part is an infinite series.}
\end{aligned}$$

8. Cycles, Homology, and Homotopy

So far, we have proved the Cauchy Integral Theorem $\int_\gamma f(z)\, dz = 0$ for (closed) paths in star-shaped domains. However, we have already seen some more general statements of a similar nature, for example most recently in the proof of Thm. 7.11, where we used that

$$\int\limits_{\partial B_{r_1}(a)} f(z)\, dz - \int\limits_{\partial B_{r_2}(a)} f(z)\, dz = 0$$

where f is holomorphic on the annulus $A_{r,R}(a)$ and $r < r_1 < r_2 < R$.

Our goal in this section is to state (in some sense) the most general form of the Cauchy Integral Theorem.

First we need a notion of how often a path winds around a given point.

8.1. **Lemma and Definition.**
Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path and $a \in \mathbb{C} \setminus \gamma := \mathbb{C} \setminus \gamma([\alpha, \beta])$. Then

$$n_\gamma(a) = \frac{1}{2\pi i} \int\limits_\gamma \frac{dz}{z - a} \in \mathbb{Z} .$$

$n_\gamma(a)$ is called the *winding number* of a with respect to γ.

Proof. Without loss of generality, $a = 0$. The idea behind the proof is that $\int dz/z$ gives us a logarithm, so the value is something like $\log|z| + i \arg z$. If we come back to the same point, the argument may have changed by an integral multiple of 2π.

To make this precise, consider for $t \in [\alpha, \beta]$

$$h(t) = \int_{\alpha}^{t} \frac{\gamma'(\tau)}{\gamma(\tau)} \, d\tau \, .$$

We have

$$\frac{d}{dt}\bigl(\gamma(t)e^{-h(t)}\bigr) = \Bigl(\gamma'(t) - \gamma(t)\frac{\gamma'(t)}{\gamma(t)}\Bigr)e^{-h(t)} = 0 \, ,$$

so $\gamma(t)/\gamma(\alpha) = e^{h(t)}$, and therefore $1 = \gamma(\beta)/\gamma(\alpha) = e^{h(\beta)}$. This implies that

$$\int_{\gamma} \frac{dz}{z} = \int_{\alpha}^{\beta} \frac{\gamma'(\tau)}{\gamma(\tau)} \, d\tau = h(\beta) \in 2\pi i \mathbb{Z} \, .$$

□

8.2. **Remark.** *The winding number $n_\gamma(a)$ is locally constant on $\mathbb{C} \setminus \gamma$.*

Proof. $n_\gamma(a) = 1/2\pi i \int_\gamma dz/(z - a)$ is a continuous function from $\mathbb{C} \setminus \gamma$ into the discrete set $\mathbb{Z}$. □

For our general Cauchy Integral Theorem, we have to generalize the notion of paths a little bit.

8.3. **Definition.** A *cycle* (or *1-cycle*) in $U \subset \mathbb{C}$ is a (formal) integral linear combination of closed paths in U. If $\gamma = n_1\gamma_1 + \cdots + n_k\gamma_k$ is a cycle in U (with closed paths γ_j and integers n_j) and $f : U \to \mathbb{C}$ is continuous, we set

$$\int_{\gamma} f(z) \, dz = n_1 \int_{\gamma_1} f(z) \, dz + \cdots + n_k \int_{\gamma_k} f(z) \, dz \, .$$

In particular, when $a \in \mathbb{C} \setminus (\gamma_1 \cup \cdots \cup \gamma_k)$, we have

$$n_\gamma(a) = n_1 n_{\gamma_1}(a) + \cdots + n_k n_{\gamma_k}(a) \, .$$

By definition, cycles are elements of an abelian group, the free abelian group generated by all closed paths in U. In particular, we can add and subtract cycles.

8.4. **Definition.** Let $U \subset \mathbb{C}$.

(1) A cycle γ in U is *homologous to zero* (w.r.t. U) if for all $a \in \mathbb{C} \setminus U$, we have $n_\gamma(a) = 0$.
(2) Two cycles γ, γ' in U are *homologous* (w.r.t. U) if $\gamma - \gamma'$ is homologous to zero.
(3) U is *simply connected* if all cycles in U are homologous to zero.

8.5. **Examples.** $\mathbb{C}$ is simply connected, since the condition is vacuous (there are no points outside U).

The punctured plane $\mathbb{C} \setminus \{0\}$ is not simply connected, since $n_{\gamma_0}(0) = 1$, where $\gamma_0 = \partial B_1(0)$. On the other hand, every cycle in $\mathbb{C} \setminus \{0\}$ is homologous to a multiple of γ_0: $\gamma - n_\gamma(0)\gamma_0$ is homologous to zero.

8.6. **Remark.** From the definition, it is clear that the cylces in U that are homologous to zero form a subgroup of the group of all cycles. We can therefore consider the quotient group $H_1(U)$. This group is called the *first homology group* of U. ("First", since we consider one-dimensional objects; there is a more general theory also considering higher-dimensional submanifolds.) U is simply connected iff $H_1(U) = 0$, and the example above shows that $H_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}$; the isomorphism being given by the winding number $\gamma \mapsto n_\gamma(0)$.

Our goal will be to prove that $\int_\gamma f(z)\,dz = 0$ for all holomorphic functions f on U if and only if γ is homologous to zero. However, we need another ingredient first.

8.7. **Definition.** Let $\gamma_0, \gamma_1 : [\alpha, \beta] \to U$ be two paths in $U \subset \mathbb{C}$. Then γ_0 and γ_1 are *homotopic in* U if there is a continuous map (a *homotopy* between γ_0 and γ_1) $\gamma : [0,1] \times [\alpha, \beta] \to U$ such that

(i) for all $u \in [\alpha, \beta]$, $\gamma(0, u) = \gamma_0(u)$ and $\gamma(1, u) = \gamma_1(u)$;
(ii) for all $t \in [0,1]$, $\gamma_t : u \mapsto \gamma(t, u)$ is picewise $\mathcal{C}^1$;
(iii) for all $t \in [0,1]$, $\gamma_t(\alpha) = \gamma_0(\alpha) = \gamma_1(\alpha)$ and $\gamma_t(\beta) = \gamma_0(\beta) = \gamma_1(\beta)$.

The intuition behind this notion is that we "continuously deform" γ_0 into γ_1 while staying in U and keeping the end-points fixed.

8.8. **Theorem (Homotopy Invariance of the Integral).**
Let $f : U \to \mathbb{C}$ be holomorphic, γ_0 and γ_1 two homotopic paths in U. Then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz\,.$$

Proof. Let $\gamma : [0,1] \times [\alpha, \beta] \to U$ be a homotopy between γ_0 and γ_1. We want to show that

$$[0,1] \ni t \longmapsto \int_{\gamma_t} f(z)\,dz$$

is locally constant. Then it follows (since $[0,1]$ is connected) that the map is constant, and we get the desired equality.

So let $t \in [0,1]$. We can cover (the image of) γ_t by finitely many open disks $D_0, D_1, \ldots, D_m \subset U$, centered at $\gamma(a_0), \ldots, \gamma(a_m)$, where $\alpha \le a_0 < a_1 < \cdots < a_m \le \beta$ (this is possible since the image of γ_t is compact). If $\delta \in \mathbb{R}$ is sufficiently small (in absolute value), $\operatorname{im}(\gamma_{t+\delta}) \subset D_0 \cup \cdots \cup D_m$. We choose points $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m \in [\alpha, \beta]$ such that $\gamma_t(\alpha_k), \gamma_{t+\delta}(\beta_k) \in D_{k-1} \cap D_k$, and we set $\alpha_0 = \beta_0 = \alpha$, $\alpha_{m+1} = \beta_{m+1} = \beta$. Let (in obvious notation)

$$\Gamma_k = \gamma_t|_{[\alpha_k, \alpha_{k+1}]} + \big[\gamma_t(\alpha_{k+1}), \gamma_{t+\delta}(\beta_{k+1})\big] - \gamma_{t+\delta}|_{[\beta_k, \beta_{k+1}]} - \big[\gamma_t(\alpha_k), \gamma_{t+\delta}(\beta_k)\big]\,,$$

then Γ_k is a closed path in the star-shaped domain D_k, hence (by Thm. 5.11) $\int_{\Gamma_k} f(z)\,dz = 0$. On the other hand, we have

$$\int_{\gamma_t} f(z)\,dz - \int_{\gamma_{t+\delta}} f(z)\,dz = \sum_{k=0}^{m} \int_{\Gamma_k} f(z)\,dz = 0\,.$$

This proves that $t \mapsto \int_{\gamma_t} f(z)\,dz$ is locally constant. $\square$

8.9. **Remark.** This result allows us to define $\int_\gamma f(z)\,dz$ for holomorphic f when γ is just continuous, and not necessarily piece-wise continuously differentiable — we set

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

where γ_1 is any piece-wise $\mathcal{C}^1$ path sufficiently close to γ. The theorem tells us that this does not depend on the choice of γ_1 (since all sufficiently close paths will be homotopic).

We can then extend the notion of homotopy to just continuous paths (simply drop the condition that γ_t is piece-wise $\mathcal{C}^1$). Then the theorem also holds for continuous paths.

8.10. **Remark.** If we consider *free homotopies* of *closed* paths — we require each γ_t to be closed, but we do not require the end-points to be fixed during the homotopy — then the same result holds, with essentially the same proof. This implies for example the result mentioned at the beginning of this section:
if $r < r_1 < r_2 < R$, and f is holomorphic on the annulus $A_{r,R}(a)$, then

$$\int_{\partial B_{r_1}(a)} f(z)\,dz = \int_{\partial B_{r_2}(a)} f(z)\,dz\,,$$

since there is the free ("radial") homotopy

$$[0,1]\times[0,1] \ni (t,u) \longmapsto \big((1-t)r_1 + tr_2\big)e^{2\pi i u}\,.$$

8.11. **Lemma.** *Let γ be a closed path, and let $a \in \mathbb{C}\setminus\gamma$ be in the unbounded component of $\mathbb{C}\setminus\gamma$. Then $n_\gamma(a) = 0$.*

Proof. Exercise. □

We can now proceed to prove the general Cauchy Integral Theorem. But first, we need a slight variant of the Cauchy Integral Formula.

8.12. **Lemma (Cauchy Integral Formula for Squares).** *Let f be holomorphic on the domain $U \subset \mathbb{C}$, and let $Q \subset U$ a closed square. Then for all $z \in Q^\circ$ (the interior of Q), we have*

$$f(z) = \frac{1}{2\pi i}\int_{\partial Q} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

(where ∂Q denotes, as usual, the boundary of the square with counter-clockwise orientation).

Proof. First observe that $f(\zeta)/(\zeta-z) = f(z)/(\zeta-z)+g(\zeta)$, where g is holomorphic on U. Since Q is contained in a star-shaped domain contained in U (e.g., take a slightly larger open square), we get

$$\frac{1}{2\pi i}\int_{\partial Q} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{f(z)}{2\pi i}\int_{\partial Q} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\partial Q} g(\zeta)\,d\zeta = f(z)n_\gamma(z)\,.$$

To compute the winding number, we take one of the vertices of Q as the starting and ending point of γ; then there is a radial homotopy to the circle that passes through the vertices. The winding number with respect to points in Q° does not change, by Thm. 8.8, and by the Cauchy Integral Formula for disks 5.12, the winding number is 1. □

8.13. **Theorem (General Cauchy Integral Theorem).** *Let $U \subset \mathbb{C}$ be a domain, $f : U \to \mathbb{C}$ holomorphic, and let γ be a cycle in U that is homologous to zero (w.r.t. U). Then*

$$\int_\gamma f(z)\,dz = 0\,.$$

Proof. Let $R > 0$ be so big that $\gamma \subset B_R(0)$. Then γ is also homologous to zero in $U' = B_R(0) \cap U$, and U' is bounded. We will construct an auxiliary cycle Γ in U' with the properties

(i) $n_\gamma(\zeta) = 0$ for all $\zeta \in \Gamma$;

(ii) for all $z \in \gamma$, $f(z) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$.

Given such a cycle Γ, the claim follows:

$$\int_\gamma f(z)\,dz = \frac{1}{2\pi i}\int_\gamma\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta\,dz = \frac{1}{2\pi i}\int_\Gamma f(\zeta)\Big(\int_\gamma \frac{dz}{\zeta - z}\Big)\,d\zeta = 0\,,$$

since the inner integral is $-n_\gamma(\zeta) = 0$.

In order to construct Γ, we put a square grid on $\mathbb{C}$ of mesh size $\delta < \frac{1}{\sqrt{2}}\operatorname{dist}(\gamma, \partial U')$. Let G be the set of closed grid squares that are contained in U', and set

$$\Gamma = \sum_{Q \in G} \partial Q\,,$$

where we cancel oppositely oriented edges when they both occur. We now have to check the properties (i) and (ii). (Note that G is finite, since U' is bounded.)

(i) Let $\zeta \in \Gamma$. Then ζ is on an edge of a square Q in G such that the square Q' on the other side of the edge is not in G. So there is a point in Q' that is outside U'. Therefore

$$\operatorname{dist}(\zeta, \partial U') \le \operatorname{diam}(Q') = \sqrt{2}\delta < \operatorname{dist}(\gamma, \partial U')\,.$$

This means that we can connect ζ to a point $w \in \mathbb{C} \setminus U'$ by a straight line segment that does not intersect γ. So $n_\gamma(\zeta) = n_\gamma(w) = 0$, the latter since γ is homologous to zero with respect to U'.

(ii) Let $z \in \gamma$. Then $z \in \bigcup_{Q \in G} Q$ (by the same inequality of distances as above). If z is not on an edge of one of the squares, then it is contained in Q_0° for exactly one $Q_0 \in G$. We obtain

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \sum_{Q \in G \setminus Q_0} \frac{1}{2\pi i}\int_{\partial Q} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_{\partial Q_0} \frac{f(\zeta)}{\zeta - z}\,d\zeta = 0 + f(z) = f(z)$$

by the Cauchy Integral Theorem for star-shaped domains 5.11 and by Lemma 8.12.

By continuity, this extends to $z \in \Big(\bigcup_{Q \in G} Q\Big)^\circ$ and hence to $z \in \gamma$.

□

There is also a general version of the Cauchy Integral Formula.

8.14. **Theorem (General Cauchy Integral Formula).** *Let $U \subset \mathbb{C}$ be a domain, $f : U \to \mathbb{C}$ holomorphic, and let γ be a cycle in U that is homologous to zero (w.r.t. U). Then for all $z \in U \setminus \gamma$, we have*

$$\frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta = n_\gamma(z) f(z)\,.$$

Proof. The function $\zeta \mapsto f(\zeta)/(\zeta - z)$ is holomorphic on $U \setminus \{z\}$. Let $\varepsilon > 0$ be so small that $\overline{B_\varepsilon(z)} \subset U$, and let $\beta = \partial B_\varepsilon(z)$. Then $\gamma - n_\gamma(z) \cdot \beta$ is homologous to zero in $U \setminus \{z\}$. By the General Cauchy Integral Theorem 8.13 and the original version of the Cauchy Integral Formula 5.12, we get

$$\begin{aligned} 0 &= \int\limits_{\gamma - n_\gamma(z)\cdot\beta} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int\limits_{\gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta - n_\gamma(z) \int\limits_{\beta} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \int\limits_{\gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta - n_\gamma(z)\, 2\pi i f(z)\,. \end{aligned}$$

□

Our next goal is the Residue Theorem, which can be understood as a far-reaching generalization of the Cauchy Integral Formula which we have just proved. But first, we need a definition.

8.15. **Definition.** Let $a \in \mathbb{C}$ be an isolated singularity of f; then f has a Laurent series expansion near a:

$$f(z) = \cdots + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \dots\,.$$

The coefficient $a_{-1} \in \mathbb{C}$ is called the *residue* of f at a; we write

$$a_{-1} = \operatorname{res}_a f = \operatorname{res}_{z=a} f(z)\,.$$

8.16. **Remark.** The relevance of the residue is explained by the following. Near a, we can write

$$f(z) = \frac{a_{-1}}{z-a} + g'(z)\,,$$

where

$$g(z) = \cdots - \frac{a_{-3}}{2(z-a)^2} - \frac{a_{-2}}{z-a} + a_0(z-a) + \frac{a_1}{2}(z-a)^2 + \dots\,.$$

If $B_\varepsilon(a)$ is a sufficiently small disk around a, then

$$\int\limits_{\partial B_\varepsilon(a)} f(z)\, dz = \operatorname{res}_a f \int\limits_{\partial B_\varepsilon(a)} \frac{dz}{z-a} + \int\limits_{\partial B_\varepsilon(a)} g'(z)\, dz = \operatorname{res}_a f \cdot 2\pi i\,.$$

This is the reason why the residue appears in the following theorem.

8.17. **Theorem (Residue Theorem).** *Let $U \subset \mathbb{C}$ be a domain, let $S \subset U$ be closed and discrete (in U; i.e., S does not have an acculumation point in U), and let $f : U \setminus S \to \mathbb{C}$ be holomorphic (so that the points in S are isolated singularities of f). Let γ be a cycle in U that is homologous to zero (w.r.t. U), such that $\gamma \cap S = \emptyset$. Then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{a \in S} n_\gamma(a)\,\mathrm{res}_a f\,,$$

and the sum has only finitely many non-zero terms.

Proof. The idea of the proof is the same as for the Cauchy Integral Formula 8.14. But first, we show the second claim. Let

$$I = \overline{\{a \in U \setminus \gamma : n_\gamma(a) \neq 0\}} = \{a \in \mathbb{C} \setminus \gamma : n_\gamma(a) \neq 0\} \cup \gamma \subset U\,,$$

then I is compact. Since $S \subset U$ is discrete and closed, the intersection $I \cap S$ is finite. This proves the second claim, since non-zero terms in the sum can only occur when $n_\gamma(a) \neq 0$.

Now, for each of the finitely many $a \in I \cap S$, let $\beta_a = \partial B_\varepsilon(a)$, where ε is so small that the corresponding closed disks are contained in U and such that $\overline{B_\varepsilon(a)} \cap S = \{a\}$. Then $\Gamma = \gamma - \sum_{a \in I \cap S} n_\gamma(a) \cdot \beta_a$ is homologous to zero in $U \setminus S$. We obtain

$$\begin{aligned} 0 = \int_\Gamma f(z)\,dz &= \int_\gamma f(z)\,dz - \sum_{a \in I \cap S} n_\gamma(a) \int_{\beta_a} f(z)\,dz \\ &= \int_\gamma f(z)\,dz - \sum_{a \in I \cap S} n_\gamma(a) \cdot 2\pi i\,\mathrm{res}_a f\,. \end{aligned}$$

□

9. Applications of the Residue Theorem

The Residue Theorem 8.17 is a very powerful tool that helps us evaluate many (real) definite integrals and also infinite series. We will give a number of sample applications.

9.1. **First Application.** Let f be a rational function without poles in $\mathbb{R}$ and such that $z^2 f(z)$ is bounded when $|z|$ is large. Then we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\mathrm{Im}(a)>0} \mathrm{res}_a f\,.$$

Note that the decay condition on f is necessary for the integral to converge. Note also that we can restrict the sum to a in the upper half-plane such that a is a pole of f.

The first example that comes to mind is

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i\,\mathrm{res}_{z=i} \frac{1}{1+z^2} = 2\pi i \frac{1}{2i} = \pi\,.$$

Here the only pole of $f(z) = 1/(1+z^2)$ in the upper half-plane is i. Note also that if a is a simple pole of f, then

$$\mathrm{res}_a f = \lim_{z \to a} (z-a) f(z)\,.$$

Maybe the second example one thinks of is

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\pi i \Big(\operatorname{res}_{z=e^{\pi i/4}} \frac{1}{1+z^4} + \operatorname{res}_{z=e^{3\pi i/4}} \frac{1}{1+z^4}\Big)$$
$$= 2\pi i \Big(\frac{1}{4e^{3\pi i/4}} + \frac{1}{4e^{9\pi i/4}}\Big) = \frac{\pi i}{2}\Big(\frac{-1-i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}\Big) = \frac{\pi}{\sqrt{2}}\,.$$

Can you generalize these to $\int_{-\infty}^{\infty} dx/(1+x^{2n})$?

Proof. To prove the claim, we consider the closed path γ_R that bounds the half-disk $B_R(0) \cap \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. If $R > 0$ is so large that $|z^2 f(z)| \le C$ for $|z| \ge R$ (in particular, all poles of f have absolute value $< R$), then by the Residue Theorem,

$$2\pi i \sum_{\operatorname{Im}(a)>0} \operatorname{res}_a f = \int_{\gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\beta_R} f(z)\,dz\,,$$

where β_R is the upper semicircle of radius R centered at 0. Note that all poles a of f in the upper half-plane are contained in the half-disk and have $n_{\gamma_R}(a) = 1$, whereas the poles in the lower half-plane have winding number zero. We now have to estimate the part of the integral we don't want, for which we use the bound on $|f|$:

$$\Big|\int_{\beta_R} f(z)\,dz\Big| \le \pi R \frac{C}{R^2} = \frac{\pi C}{R} \to 0 \quad \text{as } R \to \infty.$$

The claim now follows by letting R tend to infinity in the relation above. □

9.2. **Second Application.** We can extend the idea to include definite integrals of $f(x)\cos x$ or $f(x)\sin x$ for suitable rational functions f (with real coefficients). We need to package both versions into one.

Let f be a rational function without poles in $\mathbb{R}$ and such that $zf(z)$ is bounded for $|z|$ large. Then

$$\int_{-\infty}^{\infty} e^{ix} f(x)\,dx = 2\pi i \sum_{\operatorname{Im}(a)>0} \operatorname{res}_{z=a} e^{iz} f(z)\,.$$

Note that here, the decay condition is weaker than before. If f only tends to zero like $1/z$ as $|z| \to \infty$, then the integral does not converge absolutely, but it exists as an improper integral.

As an example, consider

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = 2\pi i \operatorname{res}_{z=i} \frac{e^{iz}}{1+z^2} = 2\pi i \frac{e^{-1}}{2i} = \frac{\pi}{e}\,.$$

Note that the imaginary part of $e^{ix}/(1+x^2)$ is an odd function, hence its integral vanishes. Would you have suspected e to show up in the answer?

Proof. We use a similar idea as in the first application. This time, we use the path $\gamma_{R,S}$ that bounds the rectangle with vertices at R, $R + (R+S)i$, $-S + (R+S)i$ and $-S$. Call the edge paths $\gamma_{R,S,1}, \ldots, \gamma_{R,S,4}$ in the indicated ordering (i.e., $\gamma_{R,S,1}$ is the oriented line segment from R to $R + (R+S)i$, etc.). If R and S are large enough so that we have $|zf(z)| \le C$ for $|z| \ge \min\{R, S\}$, then all the poles of f in the upper half-plane are inside the rectangle, and by the Residue Theorem again,

$$\begin{aligned} 2\pi i \sum_{\mathrm{Im}(a)>0} \mathrm{res}_{z=a}\, e^{iz} f(z) &= \int_{\gamma_{R,S}} e^{iz} f(z)\, dz \\ &= \int_{\gamma_{R,S,1}} e^{iz} f(z)\, dz + \int_{\gamma_{R,S,2}} e^{iz} f(z)\, dz + \int_{\gamma_{R,S,3}} e^{iz} f(z)\, dz \\ &\quad + \int_{-S}^{R} e^{ix} f(x)\, dx\,. \end{aligned}$$

Again, we have to show that the unwanted integrals vanish as $R, S \to \infty$ (independently!). For $\gamma_{R,S,1}$, we find

$$\Big|\int_{\gamma_{R,S,1}} e^{iz} f(z)\, dz\Big| = \Big|\int_0^{R+S} e^{-t} e^{iR} f(R+it)\, i\, dt\Big| \le \frac{C}{R} \int_0^{R+S} e^{-t}\, dt \le \frac{C}{R}\,,$$

and similarly $|\int_{\gamma_{R,S,3}} e^{iz} f(z)\, dz| \le C/S$. For $\gamma_{R,S,2}$, we obtain

$$\begin{aligned} \Big|\int_{\gamma_{R,S,2}} e^{iz} f(z)\, dz\Big| &= \Big|\int_{-S}^{R} e^{-(R+S)} e^{it} f(t + i(R+S))\, dt\Big| \\ &\le (R+S) e^{-(R+S)} \frac{C}{R+S} = C e^{-(R+S)}\,. \end{aligned}$$

(Note that $|z| \ge R + S$ on this path.) Hence all three integrals tend to zero as R and S tend to infinity, and the claim follows (at the same time, this shows that the integral exists). □

Note that in this proof, we exploit the fact that e^{iz} tends to zero very quickly when $\mathrm{Im}(z)$ tends to $+\infty$. This allows us to use the weaker decay condition on f.

9.3. **Extension of Second Application.** We can use a modification of the approach just discussed to include the case when f has simple poles in $\mathbb{R}$. Of course, the integral does not exist any more in this case, but we can compute its so-called *principal value.* Rather than to discuss the general case, let us look at one specific example. We compute

$$\mathrm{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx := \lim_{\varepsilon \downarrow 0} \Big(\int_{-\infty}^{-\varepsilon} \frac{e^{ix}}{x}\, dx + \int_{\varepsilon}^{\infty} \frac{e^{ix}}{x}\, dx \Big)\,.$$

We modify the path $\gamma_{R,S}$ we used previously by replacing the part $[-\varepsilon, \varepsilon]$ of $\gamma_{R,S,4}$ with the upper semicircle of radius ε centered at zero and oriented negatively (i.e., clockwise). We can estimate the integral over the three other edges as before, and letting $R, S \to \infty$, we obtain (since there are no poles in the upper half-plane)

$$0 = \int_{-\infty}^{-\varepsilon} \frac{e^{ix}}{x}\, dx - \int_{\beta_\varepsilon} \frac{e^{iz}}{z}\, dz + \int_{\varepsilon}^{\infty} \frac{e^{ix}}{x}\, dx\,,$$

where β_ε is the positively oriented semicircle. We can write $e^{iz}/z = 1/z + g(z)$, where g is holomorphic, hence

$$\int_{\beta_\varepsilon} \frac{e^{iz}}{z}\, dz = \pi i + O(\varepsilon)$$

(where the $O(\varepsilon)$ is $\le \pi\varepsilon C$, with C a bound for $|g|$ near 0). So as ε tends to zero, we find

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx = \pi i\,.$$

Taking imaginary parts, this implies

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi\,.$$

Note that this integral does exist in the usual sense. (Why can't we use $(\sin z)/z$ to evaluate the integral?)

9.4. **Third Application.** There are several other kinds of definite integrals that one can compute with the help of the Resiude Theorem, see the homework problems. Here, we want to discuss a different kind of application, which uses the theorem 'in the opposite direction' — rather than expressing integrals in terms of residues, we want to express a sum in terms of residues, with an integral as an intermediary (which goes away at the end of the process).

Let f be an even (i.e., $f(-z) = f(z)$) rational function without poles at positive integers and such that $z^2 f(z)$ is bounded for $|z|$ large. Then

$$\sum_{n=1}^{\infty} f(n) = -\frac{\pi}{2}\Big(\operatorname{res}_{z=0} f(z)\cot \pi z + \sum_{a\notin\mathbb{Z}} \operatorname{res}_{z=a} f(z)\cot \pi z\Big)\,.$$

As an example, consider the famous sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2}\operatorname{res}_{z=0} \frac{\pi\cot \pi z}{z^2}\,.$$

To find the residue, we expand $\pi \cot \pi z$ as a Laurent series at 0:

$$\begin{aligned}\pi\cot\pi z = \frac{\pi\cos\pi z}{\sin\pi z} &= \frac{1-\frac{\pi^2}{2}z^2+\dots}{z-\frac{\pi^2}{6}z^3+\dots}\\ &= \frac{1}{z}\Big(1-\frac{\pi^2}{2}z^2+\dots\Big)\Big(1+\frac{\pi^2}{6}z^2+\dots\Big) = \frac{1}{z}-\frac{\pi^2}{3}z+\dots\,.\end{aligned}$$

So

$$\frac{\pi\cot\pi z}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{3}\frac{1}{z} + \dots\,,$$

and the residue is $-\pi^2/3$. This finally gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

Proof. We note first that $\pi\cot\pi z$ has simple poles at all $n\in\mathbb{Z}$ with residue 1 and no other singularities (Exercise). Since f has no poles at nonzero integers and is even, we then have

$$\operatorname{res}_{z=\pm n} f(z)\,\pi\cot\pi z = f(n)$$

for $n\geq 1$. If N is a positive integer that is so large that $|z^2 f(z)|\leq C$ for $|z|\geq N$, and we let $\gamma_N=\partial B_{N+\frac{1}{2}}(0)$, then

$$\int_{\gamma_N} f(z)\,\pi\cot\pi z\,dz = 2\pi i\Big(2\sum_{n=1}^{N} f(n)+\operatorname{res}_{z=0} f(z)\,\pi\cot\pi z+\sum_{a\notin\mathbb{Z}}\operatorname{res}_{z=a} f(z)\,\pi\cot\pi z\Big).$$

The last sum here is over the poles of f.

Now I claim that $\cot\pi z$ is bounded if we stay away from the integers (where there are the poles). To see this, write, for $z=x+iy$,

$$\cot\pi z = i\,\frac{e^{\pi iz}+e^{-\pi iz}}{e^{\pi iz}-e^{-\pi iz}} = i\,\frac{e^{-2\pi y}e^{2\pi ix}+1}{e^{-2\pi y}e^{2\pi ix}-1}\,.$$

If $y\to+\infty$, this tends to $-i$, and if $y\to-\infty$, this tends to i. Also, the function is periodic with period 1, so these limits are uniform in x. So the function is bounded away from a horizontal strip around the real axis. On the other hand, it is also bounded on any closed rectangle whose vertical edges pass through $-1/2$ and $1/2$, with the disk $B_\delta(0)$ removed ($0<\delta<1/2$). (It is a continuous function, and the set in question is compact.) By periodicity again, we see that $\cot\pi z$ is bounded on $\mathbb{C}\setminus\bigcup_{n\in\mathbb{Z}} B_\delta(n)$. In particular, the function is bounded uniformly on γ_N, for all $N\geq 2$; let C' be a bound. Then we find

$$\Big|\int_{\gamma_N} f(z)\,\pi\cot\pi z\,dz\Big| \leq 2\pi(N+\tfrac{1}{2})\cdot\frac{C}{(N+\frac{1}{2})^2}\cdot\pi C' = \frac{2\pi^2 CC'}{N+\frac{1}{2}} \to 0 \quad \text{as } N\to\infty.$$

If we use this in the expression for the integral obtained above, the claim follows. □

9.5. **Example.** We can extend our evaluation of $\sum 1/n^2$. Let $k\geq 1$. Then

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = -\frac{1}{2}\operatorname{res}_{z=0}\frac{\pi\cot\pi z}{z^{2k}} = -\frac{1}{2}(\text{coefficient of } z^{2k-1} \text{ in } \pi\cot\pi z)\,.$$

So we have to look at the Laurent series expansion of $\pi\cot\pi z$ at zero. We have

$$\pi\cot\pi z = \pi i\frac{e^{\pi iz}+e^{-\pi iz}}{e^{\pi iz}-e^{-\pi iz}} = \pi i\frac{e^{2\pi iz}+1}{e^{2\pi iz}-1} = \pi i+\frac{1}{z}\,\frac{2\pi iz}{e^{2\pi iz}-1}\,.$$

So the coefficient we need is $(2\pi i)^{2k}=(-1)^k(2\pi)^{2k}$ times the coefficient of z^{2k} in $z/(e^z-1)$. We define numbers B_n by setting (near $z=0$)

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n\,.$$

These numbers are called *Bernoulli Numbers*; the first few are

$$B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_3=0,\ B_4=-\frac{1}{30},\ B_5=0,\ B_6=\frac{1}{42},\ B_7=0,\ \ldots\,.$$

Putting everything together, we have

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = -\frac{1}{2}\,\frac{(-1)^k(2\pi)^{2k}B_{2k}}{(2k)!} = \frac{(-1)^{k-1}2^{2k-1}\pi^{2k}B_{2k}}{(2k)!}\,.$$

For example,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2 B_2}{2} = \frac{\pi^2}{6}\,, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{-8\pi^4 B_4}{24} = \frac{\pi^4}{90}\,, \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{32\pi^6 B_6}{720} = \frac{\pi^6}{945}\,.$$

9.6. **Remark.** Bernoulli Numbers also show up when summing positive powers of n: We have

$$\sum_{n=0}^{N-1} n^k = \frac{1}{k+1} \sum_{l=0}^{k} \binom{k+1}{l} B_l N^{k+1-l} = \frac{1}{k+1} N^{k+1} - \frac{1}{2} N^k + \frac{k}{12} N^{k-1} + \dots + B_k N\,.$$

For the proof, let $S_k(N) = \sum_{n=0}^{N-1} n^k$; then

$$\sum_{k=0}^{\infty} S_k(N) \frac{z^k}{k!} = \sum_{n=0}^{N-1} e^{nz} = \frac{e^{Nz}-1}{e^z-1} = \frac{e^{Nz}-1}{z} \, \frac{z}{e^z-1} = \sum_{m=0}^{\infty} \frac{N^{m+1}}{m+1} \frac{z^m}{m!} \sum_{l=0}^{\infty} B_l \frac{z^l}{l!}\,;$$

the result then follows by expanding the product.

9.7. **Remark.** There are no similar explicit formulas for $\zeta(k) = \sum 1/n^k$ when k is odd. In fact, not much is known about these numbers. In 1977, Apéry proved that $\zeta(3)$ is irrational, and it is known that at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ must be irrational, and that $\zeta(k)$ is irrational for infinitely many positive odd integers k.

For many applications, the following observation is important. Let $f : U \to \mathbb{C}$ be meromorphic (and not the zero function); then f'/f is also meromorphic on U. This function can only have poles where f either has a pole or a zero. So let $a \in U$, and write $f(z) = (z-a)^m g(z)$ with $m \neq 0$ and g holomorphic near a and $g(a) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \frac{m(z-a)^{m-1} g(z) + (z-a)^m g'(z)}{(z-a)^m g(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}\,.$$

The second term is holomorphic near a, and so we see that f'/f has a *simple pole* at a with residue $m = \operatorname{ord}_a f$. The statement about the residue remains true if $m = 0$, even though there is no pole in this case.

9.8. **Proposition (Integral Counting Zeros and Poles).** *Let $U \subset \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$ be meromorphic. Let γ be a cycle in U that is homologous to zero (w.r.t. U), such that γ does not meet any of the zeros or poles of f. Then*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)}\, dz = \sum_{a \in U} n_\gamma(a) \operatorname{ord}_a f\,,$$

and the sum has only finitely many non-zero terms.

Proof. This follows from the Residue Theorem 8.17 and the observation just made. ∎

So we can use integrals in order to count the number of zeros minus the number of poles in some compact set.

9.9. **Remark.** Note that

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int\limits_{a}^{b}\frac{f'(\gamma(t))}{f(\gamma(t))}\,\gamma'(t)\,dt$$
$$= \frac{1}{2\pi i}\int\limits_{a}^{b}\frac{1}{(f\circ\gamma)(t)}\,(f\circ\gamma)'(t)\,dt = \frac{1}{2\pi i}\int\limits_{f\circ\gamma}\frac{dz}{z} = n_{f\circ\gamma}(0)\,.$$

So the number in question is the number of times the image path $f\circ\gamma$ winds around the origin. In this interpretation, the result above is known as the *Argument Principle:* the number of zeros minus the number of poles is $1/(2\pi)$ times the total change of argument of $f(z)$ as we follow γ.

We can use this to prove the Fundamental Theorem of Algebra another time.

9.10. **Corollary (Fundamental Theorem of Algebra).** *Let $f \in \mathbb{C}[z]$ be monic of degree n. Then f has exactly n zeros in $\mathbb{C}$ (counting multiplicity).*

Proof. We write

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = z^n\Big(1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\Big)\,.$$

Then

$$\frac{f'(z)}{f(z)} = \frac{n}{z} - \frac{a_{n-1} + \cdots + (n-1)a_1 z^{-n+2} + n a_0 z^{-n+1}}{1 + a_{n-1}z^{-1} + \cdots + a_1 z^{-n+1} + a_0 z^{-n}}\,\frac{1}{z^2} = \frac{n}{z} + h(z)\,,$$

and for $|z|$ large, the first factor of the second term is bounded, so $|h(z)| \le C/|z|^2$. Let N be the number of zeros of f, and let R be so large that the bound above holds. Then

$$N = \frac{1}{2\pi i}\int\limits_{\partial B_R(0)}\frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int\limits_{\partial B_R(0)}\frac{n}{z}\,dz + \frac{1}{2\pi i}\int\limits_{\partial B_R(0)}h(z)\,dz = n + \delta(R)$$

with

$$|\delta(R)| \le \frac{1}{2\pi}\,2\pi R\,\frac{C}{R^2} = \frac{C}{R}\,.$$

If we let R tend to infinity, we find $N = n$. (In fact, it suffices to take $R > 2C$, since $\delta(R)$ must be an integer.) □

A consequence of the Argument Principle is that functions that are sufficiently close on γ must have the same number of zeros minus poles.

9.11. **Theorem (Rouché's Theorem).** *Let f and g be meromorphic on the domain $U \subset \mathbb{C}$, and let γ be a cycle in U that is homologous to zero in U and does not pass through any (zeros or) poles of f or g. Assume that $|f(z)-g(z)| < |f(z)|$ for all $z \in \gamma$. Then*

$$\sum_{a\in U} n_\gamma(a)\,\mathrm{ord}_a f = \sum_{a\in U} n_\gamma(a)\,\mathrm{ord}_a g\,,$$

i.e., f and g have the same number of 'zeros minus poles' in the area enclosed by γ (with appropriate multiplicities).

Proof. Consider the meromorphic function $h = g/f$ on U. It is well-defined and without zeros on γ, and for $z \in \gamma$, we have

$$\left|\frac{g(z)}{f(z)} - 1\right| < 1\,,$$

so $h \circ \gamma$ is contained in $B_1(1)$. Hence 0 is contained in the unbounded component of all paths that make up γ, which implies by Lemma 8.11 that $n_{h\circ\gamma}(0) = 0$. Now we use Prop. 9.8 and Remark 9.9:

$$\begin{aligned} 0 = n_{h\circ\gamma}(0) &= \sum_{a\in U} n_\gamma(a)\operatorname{ord}_a(g/f) \\ &= \sum_{a\in U} n_\gamma(a)(\operatorname{ord}_a g - \operatorname{ord}_a f) = \sum_{a\in U} n_\gamma(a)\operatorname{ord}_a g - \sum_{a\in U} n_\gamma(a)\operatorname{ord}_a f\,. \end{aligned}$$

□

9.12. **Remark.** We can easily strengthen this result: it suffices that $g(z)$ is never a nonpositive real multiple of $f(z)$ on γ. Then $h \circ \gamma$ is contained in the slit plane $\mathbb{C} \setminus \mathbb{R}_{\le 0}$, and the origin is still in the unbounded component of the complement.

10. Sequences of Holomorphic Functions

We now want to consider sequences of holomorphic functions. We need a suitable notion of convergence that guarantees that the limit function will again be holomorphic. For continuous functions, this is the case when the convergence is uniform; in fact, it suffices to have uniform convergence locally, since continuity is a local property. It turns out that this is already sufficient for our purposes as well.

10.1. **Definition.** Let $U \subset \mathbb{C}$ be a domain, and let (f_n) be a sequence of holomorphic functions on U. We say that (f_n) is *locally uniformly* or *compactly convergent* on U if the following two equivalent conditions are satisfied.

(1) For every $a \in U$ there is $\varepsilon > 0$ such that $B_\varepsilon(a) \subset U$ and such that $(f_n|_{B_\varepsilon(a)})$ converges uniformly.

(2) For every compact subset $K \subset U$, $(f_n|_K)$ converges uniformly.

(1) implies (2), since K is covered by finitely many disks as in (1). (2) implies (1), since we can find a small open disk around a that is contained in a compact subset of U.

10.2. **Theorem (Compact Convergence of Holomorphic Functions).** *Let $U \subset \mathbb{C}$ be a domain, (f_n) a compactly converging sequence of holomorphic functions on U. Then the limit function $f = \lim_{n\to\infty} f_n$ is again holomorphic on U, and (f'_n) converges compactly to f' on U.*

Note that in order to have a similar result for real differentiable functions, we already need to assume that the derivatives converge (locally) uniformly!

Proof. Let $\Delta \subset U$ be a closed triangle. By the Cauchy Integral Theorem, we have $\int_{\partial\Delta} f_n(z)\,dz = 0$ for all n. Since (f_n) converges uniformly on Δ, this implies $\int_{\partial\Delta} f(z)\,dz = 0$. Since this holds for all triangles in U, f is holomorphic by Morera's Theorem.

For the second claim, we use the Cauchy Integral Formula. Let $a \in U$, and let $\varepsilon > 0$ such that $\overline{B_\varepsilon(a)} \subset U$ and (f_n) converges uniformly on $\overline{B_\varepsilon(a)}$. Then for $z \in \overline{B_{\varepsilon/2}(a)}$, we have

$$f_n'(z) = \frac{1}{2\pi i} \int\limits_{\partial B_\varepsilon(a)} \frac{f_n(\zeta)}{(z-\zeta)^2} \, d\zeta \,.$$

Since $1/(z-\zeta)$ is uniformly bounded (by $2/\varepsilon$) on $\partial B_\varepsilon(a)$ and (f_n) converges uniformly there, the right hand side converges uniformly (on $\overline{B_{\varepsilon/2}(a)}$) to the analogous integral for f, i.e., to $f'(z)$. □

For infinite series of holomorphic functions, there is a sufficient (but not necessary) criterion analogous to the "Weierstrass M-test".

10.3. **Definition.** Let $U \subset \mathbb{C}$ be a domain. A series $\sum_{n=0}^{\infty} f_n$ of holomorphic functions on U is *normally* or *compactly absolutely convergent* if for every comapct subset $K \subset U$, we have $\sum_{n=0}^{\infty} |f_n|_K < \infty$, where $|f|_K = \max\{|f(z)| : z \in K\}$.

10.4. **Remarks.**

(1) A normally convergent series of holomorphic functions is compactly convergent (i.e., its sequence of partial sums is). In particular, the limit function is holomorphic, and we can compute derivatives of any order term by term. The series of derivatives converges again normally (Exercise — compare the proof of Thm. 10.2).
(2) The terms in a normally convergent series can be arbitrarily re-ordered without affecting normal convergence and without changing the limit function.

10.5. **Example.** As a special case, we find again that a power series defines a holomorphic function in its disk of convergence.

10.6. **Example.** For $z \in \mathbb{C} \setminus \{0, 1\}$, consider

$$f(z) = \sum_{w \in \mathbb{C} : e^w = z} \frac{1}{w^2} \,.$$

Let $z_0 \in \mathbb{C} \setminus \{0, 1\}$. Let $\varepsilon > 0$ so that $0, 1 \notin B_{2\varepsilon}(z_0)$. If we pick $w_0 \in \mathbb{C}$ such that $e^{w_0} = z_0$, then there is a unique holomorphic logarithm function ℓ on $B_{2\varepsilon}(z_0)$ (i.e., $z = e^{\ell(z)}$) such that $\ell(z_0) = w_0$, and we can write our series there as

$$f(z) = \sum_{k \in \mathbb{Z}} \frac{1}{(\ell(z) + 2\pi i k)^2} \,.$$

On $\overline{B_\varepsilon(z_0)}$, ℓ is bounded, hence there is a constant $C > 0$ such that

$$\left| \frac{1}{(\ell(z) + 2\pi i k)^2} \right| \le \frac{C}{k^2}$$

for $|k| \gg 0$ and all $z \in \overline{B_\varepsilon(z_0)}$. Therefore the series is a normally convergent series of holomorphic functions, and f is holomorphic on $\mathbb{C} \setminus \{0, 1\}$.

10.7. **Theorem (Number of Zeros of the Limit Function).** *Let $U \subset \mathbb{C}$ be a domain, and let (f_n) be a compactly convergent sequence of holomorphic functions on U, with limit function f. If $K \subset U$ is compact such that f has no zero on ∂K, then f has exactly as many zeros in the interior K° as f_n for $n \gg 0$.*

Of course, the analogous result holds for the number of points where f takes some other fixed value $a \in \mathbb{C}$ (just consider $(f_n - a)$).

Proof. Since f does not vanish on ∂K, it is not the zero function, hence its zeros in K° are isolated and finite in number. Let $\varepsilon > 0$ such that the ε-disks around the zeros $a_1, \dots, a_k$ of f are pairwise disjoint and contained in K. Then f is bounded away from zero on the compact set $K \setminus (B_\varepsilon(a_1) \cup \dots \cup B_\varepsilon(a_k))$, so the same is true for f_n if n is large, so all the zeros of the f_n (for $n \gg 0$) lie in the union of the disks. This implies

$$\begin{aligned}\sum_{a \in K^\circ} \operatorname{ord}_a f &= \sum_{j=1}^{k} \frac{1}{2\pi i} \int\limits_{\partial B_\varepsilon(a_j)} \frac{f'(z)}{f(z)}\, dz \\ &= \sum_{j=1}^{k} \lim_{n\to\infty} \frac{1}{2\pi i} \int\limits_{\partial B_\varepsilon(a_j)} \frac{f_n'(z)}{f_n(z)}\, dz = \lim_{n\to\infty} \sum_{a \in K^\circ} \operatorname{ord}_a f_n \,.\end{aligned}$$

□

Note that the zeros of f do not accumulate in U unless f is the zero function. So unless $f = 0$, there will be many choices for K. So what this result says is that either the limit function is zero, or else the number of zeros is preserved in the limit.

10.8. **Corollary (Preservation of Injectivity).** *Let $U \subset \mathbb{C}$ be a domain, and let (f_n) be a compactly convergent sequence of injective holomorphic functions on U. Then the limit function is either constant or also injective.*

Proof. Let $f = \lim_{n\to\infty} f_n$ be the limit function. Assume that f is not constant, and let $a \in \mathbb{C}$. If f takes the value a at two distinct points in U, then we can find a compact subset K of U containing these two points and such that f does not take the value a on ∂K (for example, take the union of two sufficiently small disks centered at the two points in question). But this would imply, by Thm. 10.7 (applied to $f - a$), that f_n must also take the value a at least twice when $n \gg 0$, contradicting the assumptions. □

We will come back to general results on sequences or families of holomorphic functions soon. However, we want to look at some more specific case first.

11. The Mittag-Leffler Theorem

If f is holomorphic on $\mathbb{C}$ with the exception of isolated singularities, then at each singularity a, f has a principal part $\sum_{n=1}^{\infty} c_n (z-a)^{-n}$. The question we will study in this section is, to what extent can this be inverted: given principal parts, can we find a suitable function f, and how much freedom do we have for such an f?

The last question is easy to answer.

11.1. **Proposition.** *Let $A \subset \mathbb{C}$ be discrete and closed, and let $f, g : \mathbb{C} \setminus A \to \mathbb{C}$ be holomorphic. Then f and g have the same principal parts at every $a \in A$ if and only if $f - g$ is entire (i.e., all the singularities are removable).*

Proof. This is clear, since an isolated singularity is removable if and only if the prinicpal part there is zero. □

So the interesting part is the existence question.

11.2. **Theorem (Mittag-Leffler).** *Let $A \subset \mathbb{C}$ be discrete and closed, and for each $a \in A$, let $p_a(z) = \sum_{n=1}^{\infty} c_{a,n}(z-a)^{-n}$ be a Laurent series converging on $\mathbb{C} \setminus \{a\}$. Then there is a holomorphic function $f : \mathbb{C} \setminus A \to \mathbb{C}$ such that for all $a \in A$, the principal part of f at a is p_a.*

Proof. We can assume that A is infinite; otherwise we can just take $f = \sum_{a \in A} p_a$. Since A is discrete, it is countable, and we can write $A = \{a_1, a_2, \dots\}$; then $|a_n| \to \infty$ as $n \to \infty$. We can assume that $0 \notin A$ (otherwise we just add p_0 to our result). Then for each n, there is a Taylor polynomial P_{a_n} of p_{a_n} such that $|p_{a_n} - P_{a_n}|_{B_{|a_n|/2}(0)} \le 2^{-n}$. (Note that the Taylor series converges uniformly on closed disks contained in $B_{|a_n|}(0)$.) We claim that the series $\sum_{n=1}^{\infty}(p_{a_n} - P_{a_n})$ converges normally on $\mathbb{C} \setminus A$. To see this, let $K \subset \mathbb{C} \setminus A$ be compact. Then K is bounded, and since $|a_n| \to \infty$, there is some N such that $|a_n| > 2|z|$ for all $z \in K$ and all $n \ge N$. By construction, $|p_{a_n} - P_{a_n}|_K \le 2^{-n}$ for all such n, hence

$$\sum_{n=1}^{\infty} |p_{a_n} - P_{a_n}|_K \le \sum_{n=1}^{N-1} |p_{a_n} - P_{a_n}|_K + \sum_{n=N}^{\infty} 2^{-n} < \infty\,.$$

So $f = \sum_{n=1}^{\infty}(p_{a_n} - P_{a_n})$ is holomorphic on $\mathbb{C} \setminus A$. It remains to check that f has the correct principal parts. To this end, consider $a_k \in A$, and let

$$f_k = \sum_{n \ne k}(p_{a_n} - P_{a_n}) = f - (p_{a_k} - P_{a_k})\,.$$

Then $f = (f_k - P_{a_k}) + p_{a_k}$, where the first term is holomorphic near a_k, which implies that p_{a_k} is the principal part of f at a_k. □

11.3. **Remark.** The Mittag-Leffler Theorem holds more generally for any domain $U \subset \mathbb{C}$: if $A \subset U$ is discrete and closed in U, and for each $a \in A$, p_a is a principal part at a as above, then there is $f : U \setminus A \to \mathbb{C}$ holomorphic such that f has the prescribed principal parts.

For the proof, one uses a similar idea. We can assume that A is bounded (otherwise we map everything by $z \mapsto \frac{1}{z-b}$ where $b \in U \setminus A$; of course, we also have to map the principal parts accordingly. Since A is discrete and closed in U, b is not an accumulation point of A, hence the image of A is bounded). Writing again $A = \{a_1, a_2, \dots\}$, there is a sequence (b_n) of points in ∂U such that $|b_n - a_n| \to 0$ as $n \to \infty$. (Otherwise A would be contained in a compact set contained in U, hence A would be finite.) The role of the polynomials in the proof above is now taken by finite Laurent series centered at the b_n: By Thm. 7.11, there is a Laurent series centered at b_n that represents p_{a_n} on $\mathbb{C} \setminus \overline{B_{|b_n - a_n|}(b_n)}$ and converges uniformly on $B_n = \mathbb{C} \setminus B_{2|b_n - a_n|}(b_n)$. Let P_{a_n} be a partial sum of this series such that $|P_{a_n} - p_{a_n}|_{B_n} \le 2^{-n}$. Then

$$f(z) = \sum_{n=1}^{\infty}(p_{a_n} - P_{a_n})$$

converges normally on $U \setminus A$ (note that every compact subset $K \subset U$ is contained in all B_n for $n \gg 0$, since K has positive distance from ∂U) and has the desired principal parts.

11.4. **Remark.** In the proof, we used a bound $|p_{a_n} - P_{a_n}|_{K_n} \le 2^{-n}$. However, all that matters is that for all compact subsets K,

$$\sum_{n=1}^{\infty} |p_{a_n} - P_{a_n}|_K < \infty\,.$$

For example, this is the case when $|p_{a_n} - P_{a_n}|_K = O(n^{-2})$.

11.5. **Example.** In the homework, you prove that

$$f(z) = \frac{1}{z} + \sum_{n \in \mathbb{Z}\setminus\{0\}} \left(\frac{1}{z-n} + \frac{1}{n}\right)$$

converges normally on $\mathbb{C} \setminus \mathbb{Z}$. This is an example of the construction used in the proof (with a bound $O(n^{-2})$); it gives a meromorphic function on $\mathbb{C}$ with simple poles of residue 1 at the integers. Since $\pi \cot \pi z$ has the same poles and residues (as we have seen earlier, compare 9.4), we know that $f(z) - \pi \cot \pi z$ is an entire function. To show that actually $f(z) = \pi \cot \pi z$, we observe that the difference is bounded (see homework), hence constant by Liouville's Theorem 5.19. Finally, the constant is zero since both functions are odd.

This gives the following (the second series is obtained by grouping the terms for n and $-n$).

Theorem.

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \in \mathbb{Z}\setminus\{0\}} \left(\frac{1}{z-n} + \frac{1}{n}\right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}\,,$$

and the series converge normally on $\mathbb{C} \setminus \mathbb{Z}$.

Expanding the terms in the sum as Taylor series at the origin, we can deduce again the formulas for $\sum_{n=1}^{\infty} n^{-2k}$.

We can differentiate the series term by term and obtain:

Corollary.

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}\,.$$

In fact, it is easier to show this relation first (the series obviously converges normally on $\mathbb{C} \setminus \mathbb{Z}$, both sides have the same principal parts, are 1-periodic and tend to zero as $|\operatorname{Im}(z)| \to \infty$) and derive the expression for the cotangent from it: the difference between $f(z)$ and $\pi \cot \pi z$ must be a constant, hence zero (again because the functions are odd).

12. Infinite Products of Functions

In this section, we will develop a theory of infinte products similar to that of infinite series. There is a small catch, however, related to the special role of zero. Since we don't want convergence of a product to depend on finitely many factors, we have to disregard finitely many zero factors. This leads to the following definition.

12.1. **Definition.** An infinite product $\prod_{n=1}^{\infty} a_n$ of complex numbers *converges* if there is some $N \geq 1$ such that the limit of partial products

$$a = \lim_{n\to\infty} \prod_{k=N}^{n} a_k$$

exists and is non-zero. In this case, the value of the product is

$$\prod_{n=1}^{\infty} a_n = \left(\prod_{n=1}^{N-1} a_n\right) \cdot a\,.$$

It is zero if and only if some of the factors vanish.

12.2. **Remarks.**

(1) If $\prod_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 1$.
(2) Assume that $|a_n - 1| < 1$ for $n \geq N$. Then

$$\prod_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=N}^{\infty} \log a_n \text{ converges}\,,$$

where log denotes the principal branch of the logarithm defined on $B_1(1)$ by the usual power series

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots\,.$$

Working with a series is particularly simple when the series converges absolutely. we can use the second part of the preceding remark to carry over this notion to infinite products.

12.3. **Definition.** The infinite product $\prod_{n=1}^{\infty} a_n$ *converges absolutely* if it converges and $\sum_{n=N}^{\infty} |\log a_n| < \infty$, where N is such that $|a_n - 1| < 1$ for $n \geq N$.

There is a fairly simple criterion for absolute convergence.

12.4. **Lemma.** *The product $\prod_{n=1}^{\infty}(1 + a_n)$ converges absolutely if and only if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

Proof. Convergence on either side implies $a_n \to 0$, so without loss of generality we can assume that $|a_n|$ is so small that $\frac{1}{2}|a_n| < |\log(1 + a_n)| < \frac{3}{2}|a_n|$. (Note that $(\log(1 + z))/z \to 1$ as $z \to 0$.) This shows that

$$\sum_{n=1}^{\infty} |a_n| < \infty \iff \sum_{n=1}^{\infty} |\log(1 + a_n)| < \infty\,,$$

and the latter means that the product converges absolutely. □

Now we consider infinte products of holomorphic functions.

12.5. **Definition.** Let $U \subset \mathbb{C}$ be a domain, and let (f_n) be a sequence of holomorphic functions on U.

(1) The product $\prod_{n=1}^{\infty}(1+f_n(z))$ is *compactly convergent* if for every compact subset $K \subset U$, there is $N \geq 1$ (depending on K) such that the sequence of partial products $\prod_{k=N}^{n}(1+f_k(z))$ converges uniformly on K to a function without zeros in K.
(2) The product $\prod_{n=1}^{\infty}(1+f_n(z))$ is *normally convergent* if $\sum_{n=1}^{\infty} f_n(z)$ converges normally.

12.6. **Remark.** In a normally convergent product of holomorphic functions the terms can be arbitrarily re-ordered without affecting normal convergence or changing the limit function. Also, all products formed with a subset of the factors are again normally convergent.

12.7. **Lemma.** *Let f_n be holomorphic functions on the domain $U \subset \mathbb{C}$ and assume that $f(z) = \prod_{n=1}^{\infty}(1+f_n(z))$ converges compactly/normally, such that f does not vanish on U. Then*

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{f_n'(z)}{1+f_n(z)} \,,$$

and the series converges compactly/normally.

Proof. Write $f(z) = \prod_{n=1}^{N-1}(1+f_n(z)) \cdot F_N(z)$; then $F_N(z)$ converges compactly to 1 as $N \to \infty$. We have

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{N-1} \frac{f_n'(z)}{1+f_n(z)} + \frac{F_N'(z)}{F_N(z)} \,,$$

and F_N'/F_N converges compactly to zero. This proves the "compact convergence" version of the statement. For the "normal convergence" version, we note that $1+f_n$ is uniformly close to 1 for $n \gg 0$ and that $\sum_n f_n'$ converges normally, compare Remark 10.4. □

12.8. **Example.** The product

$$f(z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

converges normally on $\mathbb{C}$ (for $|z| \leq R$ bounded, $\sum |z^2/n^2| \leq \pi^2 R^2/6$). We compute the logarithmic derivative:

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-2z/n^2}{1-z^2/n^2} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} = \pi \cot \pi z$$

(see Example 11.5). The latter is the logarithmic derivative of $\sin \pi z$, hence $f(z) = \lambda \sin \pi z$ with some constant $\lambda \in \mathbb{C}^{\times}$. Comparison of $f'(0) = \pi$ with the derivative of $\sin \pi z$ at zero gives $\lambda = 1$:

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) .$$

The product in the preceding example was constructed in such a way as to have simple zeros at the integers. This can be generalized: given any set of prescribed zeros (without accumulation points) and mutliplicities, there is a holomorphic function that vanishes precisely at the given points with the given multiplicities. These functions are constructed as *Weierstrass Products.*

12.9. **Theorem (Weierstrass).** *Let $A \subset \mathbb{C}$ be closed and discrete (i.e., without accumulation point), and let $n : A \to \mathbb{Z}_{>0}$ be some function. Then there exists an entire function with zeros exactly at all $a \in A$ of multiplicity $n(a)$. Every such function has the form*

$$f(z) = e^{g(z)} z^{n_0} \prod_{a \in A \setminus \{0\}} \left(\left(1 - \frac{z}{a}\right) e^{\frac{z}{a} + \frac{1}{2}\left(\frac{z}{a}\right)^2 + \cdots + \frac{1}{m(a)}\left(\frac{z}{a}\right)^{m(a)}} \right)^{n(a)},$$

where g is an entire function, $n_0 = 0$ if $0 \notin A$ and $n_0 = n(0)$ if $0 \in A$, and $m : A \to \mathbb{Z}_{\geq 0}$ is a suitable function such that the product converges normally.

Proof. For existence, we show that we can define a function m as above such that the product converges normally; it is then clear that f has the required zeros.

We can assume that A is infinite (otrherwise the claim is trivial, with $m = 0$), then A is countable: $A = \{a_1, a_2, \dots\}$, and we have $|a_n| \to \infty$. We can also assume that $0 \notin A$. Let $a \in A$, then for $|z| \leq |a|/2$,

$$f_{a,m}(z) = \left(1 - \frac{z}{a}\right) \exp\left(\frac{z}{a} + \frac{1}{2}\left(\frac{z}{a}\right)^2 + \cdots + \frac{1}{m}\left(\frac{z}{a}\right)^m\right)$$

converges uniformly to 1 as $m \to \infty$. We can therefore take $m(a_n)$ such that

$$\sum_{n=1}^{\infty} \left| f_{a_n, m(a_n)}^{n(a)} - 1 \right|_{B_0(|a_n|/2)} < \infty .$$

With such a choice of $m : A \to \mathbb{Z}_{\geq 0}$, the product converges normally.

For the second statement, let f be a function with the required properties, and let $\tilde{d}$ be f divided by the product as constructed above. Then $\tilde{f}$ extends to an entire function without zeros, hence $\tilde{f}(z) = e^{g(z)}$ for some entire function g (compare homework). ☐

12.10. **Remarks.**

(1) An alternative proof is possible using the Mittag-Leffler Theorem. There is a meromorphic function with simple poles at all $a \in A$ of residue $n(a)$. This function has a logarithmic antiderivative (since the residues are integers), which is an entire function (since the residues are nonnegative) satisfying the requirements. This proof extends to arbitrary simply connected domains.

(2) In fact, Weierstrass' Theorem holds for arbitrary domains in $\mathbb{C}$.

12.11. **Corollary.** *Every meromorphic function on $\mathbb{C}$ (or on a domain U, given the general version of the theorem) is a quotient of two holomorphic functions.*

Proof. Let f be meromorphic. By the theorem, we can construct a holomorphic function g with zeros at the poles of f of order the pole order of f. Then fg extends to a holomorphic function h, so $f = h/g$ is a quotient of holomorphic functions. ☐

In algebraic terms, this says *the field of meromorphic funtions on $\mathbb{C}$ (or U) is the field of fractions of the ring of holomorphic functions on $\mathbb{C}$ (U).*

12.12. **Corollary.** *Given a closed and discrete set $A \subset \mathbb{C}$ and a map $v : A \to \mathbb{C}$, there is an entire function f such that $f(a) = v(a)$ for all $a \in A$.*

Proof. By Thm. 12.9, there is an entire function h having simple zeros in A. By Thm. 11.2, there is a meromorphic function g on $\mathbb{C}$ with simple poles at all $a \in A$ and residues $\operatorname{res}_a g = v(a)/h'(a)$. Then $f = gh$ satisfies the requirements. □

This "Interpolation Theorem" can be extended in an obvious way to obtain a function whose Taylor series at $a \in A$ matches given polynomials up to given (finite) order.

13. Montel's Theorem

We come back to the study of sequences or families of functions.

The following definition and result are valid in a more general context.

13.1. **Definition.** Let $X \subset \mathbb{R}^r$, $Y \subset \mathbb{R}^s$ be open, and let $\mathcal{F}$ be a family of functions $X \to Y$.

(1) $\mathcal{F}$ is *equicontinuous* (on X) if

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x_1, x_2 \in X \, \forall f \in \mathcal{F} : |x_1 - x_2| < \delta \Longrightarrow |f(x_1) - f(x_2)| < \varepsilon \,.$$

(2) $\mathcal{F}$ is *locally equicontinuous* (on X) if every $x \in X$ has a neighborhood U in X such that $\mathcal{F}$ is equicontinuous on U.

(3) $\mathcal{F}$ is *normal* if every sequence in $\mathcal{F}$ has a compactly convergent subsequence.

(4) $\mathcal{F}$ is *locally bounded* (on X) if for every $x \in X$ there is a neighborhood U of x in X and $B \geq 0$ such that $|f|_U \leq B$ for all $f \in \mathcal{F}$. Equivalently,

$$\forall K \subset X \text{ compact } \exists B \geq 0 \, \forall f \in \mathcal{F} : |f|_K \leq B \,.$$

(5) $\mathcal{F}$ is *point-wise bounded* if

$$\forall x \in X \, \exists B \geq 0 \, \forall f \in \mathcal{F} : |f(x)| \leq B \,.$$

13.2. **Theorem (Arzelà-Ascoli).** *If $\mathcal{F}$ is a locally equicontinuous and point-wise bounded family of functions $\mathbb{R}^r \supset X \to Y \subset \mathbb{R}^s$, then $\mathcal{F}$ is normal.*

Proof. Let (f_n) be a sequence in $\mathcal{F}$. We have to show that it has a subsequence that converges compactly. In a first step, we obtain a subsequence that converges *point-wise* on a *dense* subset of X. We choose a subset $A \subset X$ that is countable and dense (for example, all points in X with rational coordinates); let $A = \{a_1, a_2, \dots\}$. Now we recursively construct subsequences $(f_{j,n})$ of (f_n) such that $f_{0,n} = f_n$, $(f_{j+1,n})$ is a subsequence of $(f_{j,n})$, and $f_{j,n}(a_k)$ converges (as $n \to \infty$) for all $k \leq j$. This is possible since the sequence $(f_{j,n}(a_{j+1}))$ is bounded by assumption, hence we can select a convergent subsequence. The "diagonal sequence" $(g_n) = (f_{n,n})$ then will be a subsequence of (f_n) that converges point-wise on A.

Now we show that (g_n) converges compactly. Let $K \subset X$ be compact and $\varepsilon > 0$. We need to show that there is N such that $|g_n - g_m|_K < \varepsilon$ for $m, n \geq N$. We can assume that $K = \overline{B_r(x)}$ for some $x \in X$, $r > 0$. Since $\mathcal{F}$ is locally equicontinuous, there is $\delta > 0$ such that

$$\forall x_1, x_2 \in K \, \forall n : |x_1 - x_2| < \delta \Longrightarrow |g_n(x_1) - g_n(x_2)| < \frac{\varepsilon}{3} \,.$$

By our choice of K, $A \cap K$ is dense in K (for this, we need that K is the topological closure of its interior), so we can cover K by balls $B_\delta(a)$ with $a \in S \subset A \cap K$ and $\#S < \infty$. Since (g_n) converges point-wise on S and S is finite, there is N such that

$$\forall a \in S \; \forall m, n \geq N : |g_n(a) - g_m(a)| < \frac{\varepsilon}{3} .$$

Now let $x \in K$, $m, n \geq N$. There is $a \in S$ such that $|x - a| < \delta$. Then

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(a)| + |g_n(a) - g_m(a)| + |g_m(a) - g_m(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon ,$$

and therefore $|g_n - g_m|_K < \varepsilon$ as well. □

Now we apply this to holomorphic functions. As usual, we can get by with far weaker assumptions.

13.3. **Theorem (Montel).** *Let $U \subset \mathbb{C}$ be a domain. If $\mathcal{F}$ is a locally bounded family of holomorphic functions on U, then $\mathcal{F}$ is normal.*

Proof. We want to apply the Arzelà-Ascoli Theorem, so we have to show that $\mathcal{F}$ is locally equicontinuous (point-wise boundedness is clear, since $\mathcal{F}$ is locally bounded). For this, we use again the Cauchy Integral Formula in a by now familiar way. Let $a \in U$ and pick $r > 0$ such that $\overline{B_{2r}(a)} \subset U$. For $f \in \mathcal{F}$ and $w, z \in B_r(a)$, we have

$$f(z) - f(w) = \frac{1}{2\pi i} \int\limits_{\partial B_{2r}(a)} f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) d\zeta = \frac{z - w}{2\pi i} \int\limits_{\partial B_{2r}(a)} \frac{f(\zeta)}{(\zeta - z)(\zeta - w)} \, d\zeta .$$

Since $|\zeta - z|, |\zeta - w| > r$ for z, w, ζ as above, we find that

$$|f(z) - f(w)| \leq \frac{|z - w|}{2\pi} 4\pi r \frac{B}{r^2} = \frac{2B}{r} |z - w| ,$$

where B is a bound for $|f|_{\overline{B_{2r}(a)}}$ for $f \in \mathcal{F}$ (here we use the assumption that $\mathcal{F}$ is locally bounded). We get equicontinuity on $B_r(a)$ if we take $\delta = \varepsilon r/(2B)$. The Arzelà-Ascoli Theorem 13.2, applied to $\mathcal{F}$ (where $X = U \subset \mathbb{C} \cong \mathbb{R}^2$ and $Y = \mathbb{C} \cong \mathbb{R}^2$), then proves the claim. □

If (a_n) is a bounded sequence in $\mathbb{R}^r$ and every convergent subsequence of (a_n) has the same limit a, then (a_n) itself converges to a. We have a similar statement for functions.

13.4. **Corollary.** *Let (f_n) be a locally bounded sequence of holomorphic functions on a domain $U \subset \mathbb{C}$. If every compactly convergent subsequence of (f_n) has the same limit function f, then (f_n) itself converges compactly to f.*

Proof. Assume that (f_n) does not converge compactly to f. Then there is a compact subset $K \subset U$ and $\varepsilon > 0$ such that for a subsequence (f_{n_j}) of (f_n), we have $|f_{n_j} - f|_K > \varepsilon$ for all j. By Montel's Theorem 13.3, (f_{n_j}) has a subsequence that converges compactly, and by assumption, the limit function must be f. This contradicts the conclusion we drew from the assumption that (f_n) does not converge compactly to f, hence this assumption must be false. □

Combining this with the uniqueness theorem 6.9, we obtain the following result.

13.5. **Theorem (Vitali).** *Let (f_n) be a locally bounded sequence of holomorphic functions on a domain $U \subset \mathbb{C}$. Assume that the set*

$$A = \{z \in U : \lim_{n\to\infty} f_n(z) \text{ exists}\}$$

has an accumulation point in U. Then (f_n) converges compactly on U. The same conclusion holds when there is a point $a \in U$ such that $f_n^{(k)}(a)$ converges (as $n \to \infty$) for all $k \geq 0$.

Proof. Let (g_n) and (h_n) be two subsequences of (f_n) that converge compactly, with limit functions g and h, respectively. By assumption, g and h agree on A. Theorem 6.9 then implies that $g = h$ on U. So all compactly convergent subsequences of (f_n) have the same limit function, hence by the preceding result, (f_n) converges compactly.

For the second statement, the proof is similar, using part (ii) of Theorem 6.9. □

13.6. **Example.** We show how to use Vitali's Theorem in order to show convergence of a sequence of functions from very limited knowledge. Let

$$f_n(z) = \left(1 + \frac{z}{n}\right)^n .$$

We know that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} \exp\left(n \log\left(1 + \frac{1}{n}\right)\right) = \exp\left(\lim_{n\to\infty} n \log\left(1 + \frac{1}{n}\right)\right) = e\,.$$

This implies that

$$\lim_{n\to\infty} f_n\left(\frac{1}{k}\right) = e^{1/k}$$

converges for all $k \geq 1$. We also know that

$$|f_n(z)| \leq \left(1 + \frac{|z|}{n}\right)^n = \exp\left(n \log\left(1 + \frac{|z|}{n}\right)\right) \leq \exp\left(n \frac{|z|}{n}\right) = e^{|z|}\,,$$

so (f_n) is locally bounded on $\mathbb{C}$. Theorem 13.5 now implies (since $\{1/k : k \geq 1\}$ accumulates in zero) that (f_n) converges compactly on $\mathbb{C}$, and the limit function must be e^z, since it agrees with it on $\{1/k : k \geq 1\}$.

You may remember from an early homework problem that if $U \subset \mathbb{C}$ is a domain, $\gamma : [a,b] \to \mathbb{C}$ is a continuously differentiable path, and $f : \operatorname{im}(\gamma) \times U \to \mathbb{C}$ is continuous, and $f(\gamma(t), z)$ is holomorphic in U for every fixed $t \in [a,b]$, then

$$F(z) = \int_\gamma f(\zeta, z)\, d\zeta$$

is holomorphic in U. This can be proved using Morera's Theorem, see Remark 5.6. We will now prove this again, and more, as an application of Vitali's Theorem.

13.7. **Theorem.** *Let $U \subset \mathbb{C}$ be a domain, and let $\gamma : [0,1] \to \mathbb{C}$ be a continuously differentiable path. Assume that $f : \operatorname{im}(\gamma) \times U \to \mathbb{C}$ is locally bounded (e.g., continuous), that for every $t \in [0,1]$, $z \mapsto f(\gamma(t), z)$ is holomorphic in U, and that for every $z \in U$, the Riemann integral $\int_\gamma f(\zeta, z)\, d\zeta$ exists. Then the function*

$$F(z) = \int_\gamma f(\zeta, z)\, d\zeta$$

is holomorphic in U. If in addition the Riemann integrals $\int_\gamma \frac{\partial f}{\partial z}(\zeta, z)\, d\zeta$ exist for every $z \in U$, then

$$F'(z) = \int\limits_\gamma \frac{\partial f}{\partial z}(\zeta, z)\, d\zeta\,.$$

Proof. The desired conclusion is local on U, so we can assume that f is bounded, say by B. Let $g(t, z) = f(\gamma(t), z)\gamma'(t)$, then $\int_\gamma f(\zeta, z)\, d\zeta = \int_0^1 g(t, z)\, dt$. By assumption, for every $z \in U$, the sequence of Riemann sums

$$S_n(z) = \frac{1}{n}\sum_{k=1}^{n} g\Big(\frac{k}{n}, z\Big)$$

converges to $F(z)$. Each function S_n is holomorphic, and $|S_n|_U \le B\,|\gamma'|_{[0,1]}$. By Vitali's Theorem 13.5, (S_n) converges compactly to F, hence F is holomorphic.

Under the additional hypothesis, the sequence

$$S_n'(z) = \frac{1}{n}\sum_{k=1}^{n} \frac{\partial g}{\partial z}\Big(\frac{k}{n}, z\Big)$$

converges point-wise to $\int_\gamma \frac{\partial f}{\partial z}(\zeta, z)\, d\zeta$, but also compactly to $F'(z)$. □

13.8. **Example.** Recall that for $t > 0$, we can define an entire function $t^z = e^{z \log t}$. Then by Theorem 13.7, we see immediately that for $0 < a < b$, the function

$$f_{a,b}(z) = \int\limits_a^b t^{z-1} e^{-t}\, dt$$

is holomorphic on $\mathbb{C}$. It is not hard to see that for $0 < x$ real, the integral

$$\int\limits_0^\infty t^{x-1} e^{-t}\, dt$$

converges. This implies that the family $\{f_{a,b} : 0 < a < b\}$ is locally bounded on the right half-plane $R = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$: if $0 < c \le \mathrm{Re}(z) \le d$, then

$$|f_{a,b}(z)| \le \int\limits_0^1 t^{c-1} e^{-t}\, dt + \int\limits_1^\infty t^{d-1} e^{-t}\, dt\,.$$

Vitali's Theorem 13.5 now implies that (letting $a \to 0$ and $b \to \infty$)

$$\Gamma(z) = \int\limits_0^\infty t^{z-1} e^{-t}\, dt$$

exists on R and is holomorphic there. In addition, integration by parts gives

$$z\Gamma(z) = \int\limits_0^\infty z t^{z-1} e^{-t}\, dt = \int\limits_0^\infty t^z e^{-t}\, dt = \Gamma(z+1)\,.$$

This functional equation allows us to extend Γ to a function that is holomorphic on $\mathbb{C} \setminus \mathbb{Z}_{\le 0}$.

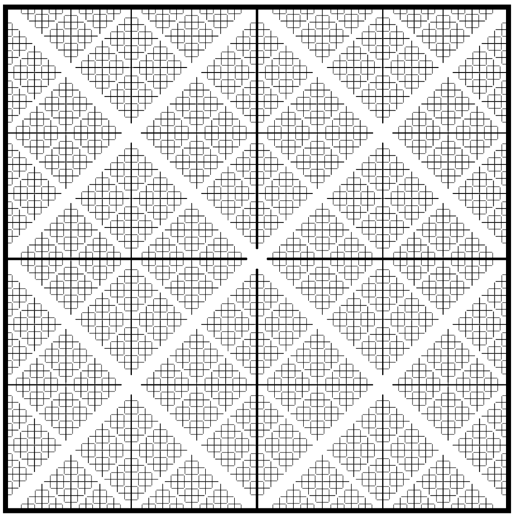
14. The Riemann Mapping Theorem

In this section, we prove one of the very famous theorems in complex analysis.

14.1. **Theorem (Riemann Mapping Theorem).** *Let $U \subsetneq \mathbb{C}$ be a simply connected domain. Then there is a biholomorphic map $f : U \to B_1(0)$.*

Note that Liouville's Theorem 5.19 implies that the conclusion cannot hold for $U = \mathbb{C}$.

This is an extremely strong result: we have learnt in this course that holomorphic functions are very 'rigid' — they are determined by a small amount of information. Yet they can be used to transform any simply connected domain to a simple open disk. Note that simply connected domains can be extremely complicated, like for example the following (really, the limiting case of the indicated recursive process):



It is hard to imagine that there can be a conformal (i.e., angle-preserving) mapping from this to the unit disk.

14.2. **Corollary.** *Every simply connected domain is homeomorphic to the open unit disk.*

Proof. Let $U \subset \mathbb{C}$ be a domain. When $U \neq \mathbb{C}$, then this follows from the Riemann Mapping Theorem 14.1, since a biholomorphic map is a homeomorphism. When $U = \mathbb{C}$, this can be checked directly, e.g., using the map $z \mapsto \tanh(|z|)\, z/|z|$. □

We will prove the theorem in several steps. First note that an injective holomorphic map automatically is biholomorphic onto its image: we have seen in Cor. 6.7 that an injective holomorphic function has non-vanishing derivative, hence the inverse function is again holomorphic. So it suffices to show that there is an injective holomorphic map from U *onto* $B_1(0)$. In a first step, we show that there are injective holomorphic maps from U *into* $B_1(0)$.

14.3. **Lemma.** *Let $U \subsetneq \mathbb{C}$ be a simply connected domain. Then there is an injective holomorphic function $f : U \to B_1(0)$.*

Proof. First, we want to map U biholomorphically to a domain whose complement contains an open ball. This need not be the case for U itself; an example for this is the slit plane $\mathbb{C} \setminus \mathbb{R}_{\le 0}$.

Let $a \in \mathbb{C} \setminus U$. The function $z - a$ does not vanish on U. Since U is simply connected, there is a "square root" s of this fuction on U, i.e., a holomorphic function $s : U \to \mathbb{C}$ such that $s(z)^2 = z - a$. This function s is injective: $s(z) = s(w)$ implies $z - a = s(z)^2 = s(w)^2 = w - a$ and hence $z = w$. Let $U_1 = s(U)$ be the image of U under s. Since non-constant holomorphic maps are open, there is $b \in U_1$ and $r > 0$ such that $B_r(b) \subset U_1$. I claim that $B_r(-b) \subset \mathbb{C} \setminus U_1$. Assume that there is $w \in U_1 \cap B_r(-b)$. Then $w = s(z)$ for some $z \in U$; also $-w = s(z')$ for some $z' \in U$ (since $-w \in B_r(b) \subset U_1$). But this implies $z = s(z)^2 + a = w^2 + a = s(z')^2 + a = z'$, so $w = s(z) = s(z') = -w$, which is a contradiction, since $0 \notin U_1$.

Now we let
$$f(z) = \frac{r}{s(z) + b};$$
then f is injective and holomorphic on U, and $f(U) \subset B_1(0)$. □

We refine the previous statement a little bit.

14.4. **Lemma.** *Let $U \subsetneq \mathbb{C}$ be a simply connected domain, and let $a \in U$. Then there is an injective holomorphic function $f : U \to B_1(0)$ such that $f(a) = 0$.*

Proof. By the previous lemma, there is $f_0 : U \to B_1(0)$ holomorphic and injective. Then
$$f(z) = \frac{1}{2}\bigl(f_0(z) - f_0(a)\bigr)$$
does what we want. □

Let now
$$\mathcal{F} = \{f : U \to B_1(0) : f \text{ holomorphic and injective, } f(a) = 0\}.$$
We have seen that $\mathcal{F}$ is non-empty. The biholomorphic map $U \to B_1(0)$ we want to find will be an element of $\mathcal{F}$. We detect it by some extremal property.

We pick some point $b \in U \setminus \{a\}$. Let
$$\rho = \sup\{|f(b)| : f \in \mathcal{F}\}.$$
The idea is that we want the image of U to fill out the unit disk, so we want to move any given point as far away from the center as possible. Note that $\rho > 0$, since $\mathcal{F}$ is non-empty, and $f(b) \neq 0$ for every $f \in \mathcal{F}$.

14.5. **Claim.** *There is some $f \in \mathcal{F}$ such that $|f(b)| = \rho$.*

Proof. By definition of ρ, there is a sequence (f_n) in $\mathcal{F}$ such that $|f_n(b)| \to \rho$ as $n \to \infty$. Also, $\mathcal{F}$ is bounded, so by Montel's Theorem 13.3, there is a compactly convergent subsequence, and without loss of generality, we can assume that (f_n) itself converges compactly. Let $f = \lim_{n\to\infty} f_n$ be the limit function. Then clearly $|f(b)| = \rho$. It remains to show that $f \in \mathcal{F}$.

It is clear that f is not constant ($f(a) = 0$, $|f(b)| = \rho > 0$). Hence by Cor. 10.8, f is injective as the limit of a sequence of injective holomorphic functions. Finally, $f(U)$ is open and contained in $\overline{B_1(0)}$, so $f(U) \subset B_1(0)$. □

Now we show that we have found the function we want.

14.6. **Lemma.** *Let $w \in B_1(0)$, and define*

$$\phi_w(z) = \frac{w - z}{1 - \bar{w}z} .$$

Then ϕ_w is an involutory automorphism of $B_1(0)$ that interchanges 0 and w.

Proof. For $z \in B_1(0)$, we have $(1-|w|^2)(1-|z|^2) > 0$, so $|w|^2 + |z|^2 < 1 + |w|^2|z|^2$. Therefore

$$|w - z|^2 = |w|^2 - \bar{w}z - w\bar{z} + |z|^2 < 1 - \bar{w}z - w\bar{z} + |w|^2|z|^2 = |1 - \bar{w}z|^2 ,$$

hence $|\phi_w(z)| < 1$. It is an easy calculation to show that $\phi_w(\phi_w(z)) = z$; this also implies that ϕ_w is an automorphism. Finally, it is clear that $\phi_w(0) = w$ and $\phi_w(w) = 0$. □

14.7. **Claim.** *The function f above satisfies $f(U) = B_1(0)$.*

Proof. Assume the claim is false. Then there is $w \in B_1(0) \setminus f(U)$. We construct another function $g \in \mathcal{F}$ such that $|g(b)| > |f(b)|$, which contradicts the choice of f.

First note that $\phi_w \circ f$ is an injective holomorphic function $U \to B_1(0)$ whose image does not contain zero. Since $U' = (\phi_w \circ f)(U)$ is simply connected, there is then a holomorphic square root function s on U', which is injective. So $s \circ \phi_w \circ f$ is an injective holomorphic function that maps U into $B_1(0)$ again. We have $(s \circ \phi_w \circ f)(a) = s(w)$, hence if we define $g = \phi_{s(w)} \circ s \circ \phi_w \circ f$, then g is an injective holomorphic function that maps U into $B_1(0)$ and such that $g(a) = 0$, so $g \in \mathcal{F}$.

We still have to show that $|g(b)| > \rho = |f(b)|$. For this, define

$$h(z) = \phi_w\big(\phi_{s(w)}(z)^2\big) ;$$

then h is a holomorphic map $B_1(0) \to B_1(0)$, which is not an automorphism (because of the squaring in the middle). Hence by the Schwarz Lemma Cor. 6.14, $|h(z)| < |z|$ for all $z \in B_1(0) \setminus \{0\}$. In particular,

$$|g(b)| > \big|h\big(g(b)\big)\big| = |f(b)|$$

(since $h \circ g = f$). This is the desired contradiction. □

14.8. **Examples.** A “Riemann map” for the upper half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is given by

$$f(z) = \frac{z - i}{z + i} .$$

For the slit plane $U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, we use the idea of the first lemma above: there is a square root function s on U; we can pick the one with $s(1) = 1$. Then $s(U)$ is the right half plane, so after a rotation, we are back to the upper half plane. So a suitable function is

$$f(z) = \frac{is(z) - i}{is(z) + i} = \frac{s(z) - 1}{s(z) + 1} .$$

15. The Riemann Sphere

It is often useful to consider the *extended complex plane* that is obtained by adding a "point at infinity" to the standard complex plane. This extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ can then serve both as the domain and as the target of holomorphic maps.

The idea is to consider the extended complex plane near infinity to look like the normal complex plane near zero, under the identification $z \mapsto 1/z$. For example, neighborhoods of the origin in $\mathbb{C}$ contain an open disk $B_\varepsilon(0)$, so neighborhoods of ∞ in $\hat{\mathbb{C}}$ should contain ∞ together with the exterior of a closed disk $\overline{B_R(0)}$. This defines the topology of $\hat{\mathbb{C}}$ (together with the obviously desirable requirement that open subsets of $\mathbb{C}$ continue to be open in $\hat{\mathbb{C}}$).

In order to define when maps from or to $\hat{\mathbb{C}}$ are holomorphic, we again use $1/z$ to transform the question back to the origin.

15.1. **Definition.** Let $U \subset \mathbb{C}$ be open and $f : U \to \hat{\mathbb{C}}$ a continuous map. Let $I = f^{-1}(\infty)$ and $Z = f^{-1}(0)$. f is *holomorphic*, if $f|_{U \setminus I}$ is holomorphic, and the map $z \mapsto 1/f(z)$ for $z \in U \setminus (I \cup Z)$ extends to a holomorphic map g on $U \setminus Z$ such that $g(z) = 0$ for $z \in I$.

If $U \subset \hat{\mathbb{C}}$ is open, then $f : U \to \hat{\mathbb{C}}$ is *holomorphic* if $f|_{U \setminus \{\infty\}}$ is holomorphic, and $z \mapsto f(1/z)$ extends to a holomorphic map on $\{z \in \mathbb{C}^\times : 1/z \in U\} \cup \{0\}$.

A nice feature of this is that we can now consider meromorphic functions as holomorphic functions into $\hat{\mathbb{C}}$.

15.2. **Proposition.** *Let $U \subset \mathbb{C}$ be a domain. Then there is a natural bijection between meromorphic functions on U and holomorphic functions $U \to \hat{\mathbb{C}}$ with the exception of the constant map $z \mapsto \infty$.*

Proof. Given $f : U \to \mathbb{C}$ meromorphic, we define $\hat{f} : U \to \hat{C}$ by $\hat{f}(z) = f(z)$ if z is not a pole of f and $\hat{f}(z) = \infty$ if z is a pole of f. We have to show that $\hat{f}$ is holomorphic. This is clear in a neighborhood of any point $a \in U$ that is not a pole of f. If $a \in U$ is a pole of f, f does not vanish on some neighborhood of a, so $z \mapsto 1/f(z)$ is defined and holomorphic on some punctured disk $B_\varepsilon(a)$. Since a is a pole of f, this function now has a removable singularity at a (which can be filled with the value 0), so $1/f$ extends to a holomorphic function on $B_\varepsilon(a)$.

Conversely, assume that $h : U \to \hat{\mathbb{C}}$ is holomorphic and not constant ∞. Then $h^{-1}(\infty)$ is discrete and closed in U. Let $f = h|_{U \setminus h^{-1}(\infty)} : U \to \mathbb{C}$. Then f is clearly holomorphic. If $a \in U$ such that $h(a) = \infty$, we have that $|f(z)| \to \infty$ as $z \to a$ (since $h(z) \to \infty$ in the topology of $\hat{\mathbb{C}}$), so a must be a pole of f. We see that f has only isolated singularities in U that are poles, so f is meromorphic on U.

Finally, it is clear that these two constructions are inverses of each other. □

15.3. **Proposition.** *Let $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be holomorphic. Then $h = \infty$ (constant function), or else $h = \hat{f}$ with a rational function f.*

Proof. To see this, note that by the preceding proposition, $h = \hat{f}$ for a meromorphic function f on $\mathbb{C}$. But we can also consider $f(1/z)$, which will extend to a meromorphic function on $\mathbb{C}$ in the same way. This shows that the poles of f

cannot accumulate at infinity, so f has only finitely many poles. Hence we can write f as the sum of a rational function that tends to zero as z tends to infinity (sum of principal parts) and an entire function g. Since $f(1/z)$ has at worst a pole at zero, we see that $f(z)$ and therefore also $g(z)$ is bounded by a polynomial in z as z tends to ∞. This implies that g is a polynomial, hence f is a rational function. □

15.4. **Corollary.** *The automorphisms of $\hat{\mathbb{C}}$ (i.e., the biholomorphic maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$) are exactly the fractional linear transformations*

$$z \longmapsto \frac{az+b}{cz+d} \qquad \text{with } ad \neq bc.$$

These maps are also called Möbius transformations.

Proof. Let $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be holomorphic and bijective. By Prop. 15.3, we have that $h = \hat{f}$ for some rational function f. Write $f(z) = p(z)/q(z)$ with coprime polynomials p and q. We want to show that p and q have degree at most 1, so assume that $\max\{\deg p, \deg q\} \geq 2$. Given $w \in \mathbb{C}$, we have that $h(z) = w$ if $wq(z) - p(z) = 0$. For all but at most one choice of w, this is a polynomial in z of degree at least 2, so it will have at least two roots, counting multiplicities. In any case, there will be a punctured neighborhood of w such that every w' in that punctured neighborhood will have at least two preimages under h. (This follows from our study of the local behavior of holomorphic functions, see for example Cor. 6.6.) But then h is not injective, a contradiction. So h must have the given form; the condition $ad \neq bc$ is then equivalent to saying that h is not constant. Conversely, it is easy to see that a fractional linear transformation is biholomorphic; indeed, the inverse of

$$z \longmapsto \frac{az+b}{cz+d} \qquad \text{is} \qquad z \longmapsto \frac{dz-b}{-cz+a}\,.$$

□

15.5. **Remark.** One can think of Möbius transformations as being given by matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(z \mapsto \frac{az+b}{cz+d} \right).$$

This induces a group homomorphism(!) $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{Aut}(\hat{\mathbb{C}})$, which is surjective by the corollary above. Its kernel consists of the scalar matrices λI_2, so we find that

$$\mathrm{Aut}(\hat{\mathbb{C}}) \cong \mathrm{PGL}_2(\mathbb{C}) = \frac{\mathrm{GL}_2(\mathbb{C})}{\mathbb{C}^\times \cdot I_2}\,.$$

15.6. **Corollary.** *The automorphisms of $\mathbb{C}$ are exactly the maps $z \mapsto az+b$ with $a \neq 0$.*

Proof. We first claim that any automorphism of $\mathbb{C}$ extends to an automorphism of $\hat{\mathbb{C}}$, which then must fix ∞. The statement then follows from the preceding corollary.

To prove the claim, let $f : \mathbb{C} \to \mathbb{C}$ be an automorphism. Consider $g : z \mapsto f(1/z)$ for $z \in B_1(0) \setminus \{0\}$. The image $f(B_1(0))$ is open (since f is not constant), so it contains a disk $B_\varepsilon(a)$. Then $z \mapsto 1/(g(z) - a)$ is bounded, hence has a removable singularity at 0. This implies that g extends to a holomorphic map $B_1(0) \to \hat{\mathbb{C}}$;

therefore f extends to a holomorphic map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which must be a bijection, hence an automorphism. □

15.7. **Corollary.** *The automorphisms of the punctured plane $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ are exactly the maps $z \mapsto az^{\pm 1}$ with $a \neq 0$.*

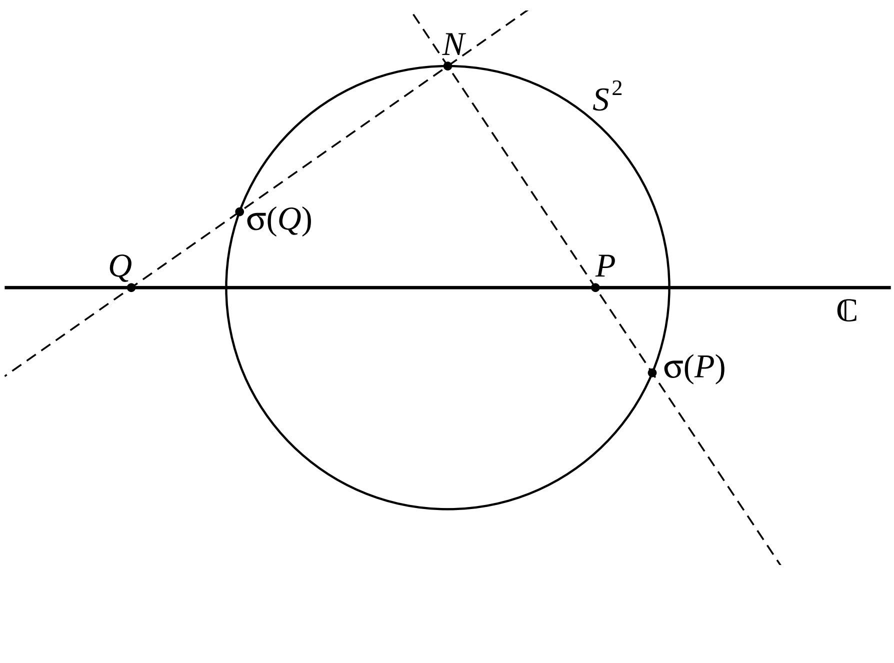
Proof. In the same way as in the preceding proof, we see that automorphisms of $\mathbb{C}^\times$ extend to automorphisms of $\hat{\mathbb{C}}$, which fix the set $\{0, \infty\}$. If the set is fixed point-wise by such an automorphism h, then the map must be of the form $z \mapsto az$ (it is $z \mapsto az + b$ and fixes zero); otherwise $z \mapsto h(1/z)$ fixes the set point-wise, and $h(z) = a/z$. □

15.8. **Corollary.** *The set $U = \hat{\mathbb{C}} \setminus \{z_1, \ldots, z_m\}$, where $z_1, \ldots, z_m$ are distinct and $m \geq 3$, has only finitely many automorphisms.*

Proof. It is not hard to see that, given $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, there is a Möbius transformation that sends z_1 to ∞, z_2 to 0, and z_3 to 1. Hence we can assume that $z_1 = \infty$, $z_2 = 0$, $z_3 = 1$. As in the previous proofs, any automorphism of U must be a Möbius transformation that fixes the set $T = \{\infty, 0, 1, z_4, \ldots, z_m\}$. Let G be the automorphism group of U; then we get a group homomorphism $G \to S_m$, where S_m is the symmetric group acting on the set T of m elements. The kernel of this homomorphism consists of the Möbius transformations fixing T point-wise. Now the only such transformations fixing 0 and ∞ are of the form $z \mapsto az$, and this fixes 1 only when $a = 1$. Hence the homomorphism has trivial kernel and so is injective. It follows that $\#G \leq m!$. □

The extended complex plane $\hat{\mathbb{C}}$ is also called the *Riemann Sphere*. The reason for this is that there is a natural identification of $\hat{\mathbb{C}}$ and the points on a two-dimensional sphere, which is given by the *stereographic projection.*

For this, we identify $\mathbb{C}$ with the xy-plane in $\mathbb{R}^3$, and we let S^2 be the unit sphere in $\mathbb{R}^3$. Denote by N the "north pole" $(0, 0, 1)$ of S^2. Then every line through N that is not parallel to the xy-plane will intersect $\mathbb{C}$ in exactly one point, and will intersect S^2 in exactly one other point besides N. In this way, we obtain a homeomorphism σ between $\mathbb{C}$ and $S^2 \setminus \{N\}$. The following picture shows this in a cross-section.



Now I claim that this extends to a homeomorphism (still called σ) between $\hat{\mathbb{C}}$ and S^2 (that sends ∞ to N). To see this, note that standard neighborhoods of ∞ in $\hat{\mathbb{C}}$ correspond exactly to standard neighborhoods of N in S^2 under σ.

In formulas, we have

$$\sigma(z) = \left(\frac{2\operatorname{Re} z}{1+|z|^2}, \frac{2\operatorname{Im} z}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2}\right) \quad \text{and} \quad \sigma^{-1}(x_1, x_2, x_3) = \frac{x_1 + i x_2}{1 - x_3}\,.$$

If we denote by $d(z, z')$, for $z, z' \in \hat{\mathbb{C}}$, the distance between $\sigma(z)$ and $\sigma(z')$ on S^2, we have

$$d(z, z') = \frac{2|z - z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}} \quad \text{and} \quad d(z, \infty) = \frac{2}{\sqrt{1+|z|^2}}$$

for $z, z' \in \mathbb{C}$ (Exercise). This "spherical metric" on $\hat{\mathbb{C}}$ is sometimes convenient, since it treats ∞ like any other point.

The spherical metric allows us, for example, to talk about compact convergence of holomorphic maps $U \to \hat{\mathbb{C}}$, which gives a notion of convergence of sequences of meromorphic functions that is more general than what we have been using so far. For example, $1/(z - 1/n)$ converges compactly to $1/z$ as $n \to \infty$ in this sense as meromorphic functions on $\mathbb{C}$, whereas this statement would not even have made much sense before. In this setting, Montel's Theorem is still valid if the assumption "locally bounded" is replaced by "locally not dense":

15.9. **Montel's Theorem for the Riemann Sphere.** *Let $U \subset \hat{\mathbb{C}}$ be non-empty, open and connected, and let $\mathcal{F}$ be a family of holomorphic maps $U \to \hat{\mathbb{C}}$. If for every compact subset $K \subset U$, the image $\bigcup_{f \in \mathcal{F}} f(K)$ is not dense in $\hat{\mathbb{C}}$, then $\mathcal{F}$ is normal: every sequence in $\mathcal{F}$ has a subsequence that converges compactly in $\hat{\mathbb{C}}$ with respect to the spherical metric.*

Proof. We have to verify that $\mathcal{F}$ is locally equicontinuous (as before in the original proof). So let $K \subset U$ be compact. Since $\bigcup_{f \in \mathcal{F}} f(K)$ is not dense in $\hat{\mathbb{C}}$, there is some $a \in \hat{\mathbb{C}}$ such that the images $f(K)$ do not meet a neighborhood of a. But then $\mathcal{F}' = \{1/(f - a) : f \in \mathcal{F}\}$ is bounded on K, so equicontinuous in the usual sense. Since $d(z, z') \le 2|z - z'|$, we have equicontinuity also with respect to the spherical metric. Finally, the map $z \mapsto a + 1/z$ is a homeomorphism $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, and $\hat{\mathbb{C}}$ is compact. Since with $g = 1/(f - a)$, we have $f = a + 1/g$, this implies that $\mathcal{F}$ is equicontinuous on K with respect to the spherical metric. ☐

15.10. **Example.** Consider the map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $z \mapsto z^2$, and let

$$\mathcal{F} = \{\mathrm{id}, f, f \circ f, f \circ f \circ f, \dots\}$$

be the iterates of f. Then $\mathcal{F}$ is normal in $B_1(0)$ (the "lower hemisphere") and also in $\hat{\mathbb{C}} \setminus \overline{B_1(0)}$ (the "upper hemisphere"), but not in any open set meeting the unit circle S^1 (the "equator"). The first statement follows from Montel's Theorem, since all iterates of f map the lower (upper) hemisphere into itself, hence the image is not dense. For the second statement, let $U \subset \hat{\mathbb{C}}$ be open and $a \in U \cap S^1$. Then some neighborhood V of a is contained in U, and some iterate of f will map V to an open set containing S^1 (V contains some arc of S^1, whose length is doubled when applying f). Iterating f further, the images will exhaust $\mathbb{C}^\times$, and so no subsequence can converge uniformly on a set containing V.

In general, if f is any holomorphic map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the set of points of $\hat{\mathbb{C}}$, where we have this "interesting" behavior is called the *Julia set* of f. More precisely, $z \in \hat{\mathbb{C}}$ is in the Julia set, if there is no neighborhood of z on which the family of iterates of f is normal. In the simple example above, where $f(z) = z^2$, the Julia set is the circle S^1. In most other cases, the Julia set is very complicated, however.

16. Picard's Theorem

Liouville's Theorem 5.19 tells us that a bounded entire function is constant. From this, one can fairly easily conclude that the image of a non-constant entire function must be dense in $\mathbb{C}$ (compare the proof of the Casorati-Weierstrass Theorem 7.7). In this section, we want to show a much stronger result: a non-constant entire function can leave out at most one value. The exponential function is an example showing that it is indeed possible that one value is missing.

We begin with a result on the existence of not too small disks in the image of a holomorphic function.

16.1. **Definition.** Let $U \subset \mathbb{C}$ be a domain. We say that a function $f : \bar{U} \to \mathbb{C}$ is *holomorphic*, if f is the restriction of a holomorphic function on an open set containing $\bar{U}$.

16.2. **Theorem (Bloch).** *Let $f : \overline{B_1(0)} \to \mathbb{C}$ be holomorphic. Assume that the function $z \mapsto |f'(z)|(1 - |z|^2)$ attains its maximum $M > 0$ at the point $q \in B_1(0)$. Then $f(B_1(0))$ contains the open disk of radius $\left(\frac{3}{2}\sqrt{2} - 2\right)M$ with center $f(q)$. In particular, if $f'(0) = 1$, then $f(B_1(0))$ contains an open disk of radius $\frac{3}{2}\sqrt{2} - 2 > 0$.*

For the proof, we will use the following lemma.

16.3. **Lemma.** *Let f be a non-constant holomorphic function on $\overline{B_r(0)}$ such that $|f'|_{\overline{B_r(0)}} \le 2|f'(0)|$. Then $f(B_r(0))$ contains the open disk around $f(0)$ of radius $R = (3 - 2\sqrt{2})r|f'(0)|$.*

Proof. Without loss of generality, $f(0) = 0$. Let $g(z) = f(z) - f'(0)z$. Then we can write

$$g(z) = \int_0^z \big(f'(\zeta) - f'(0)\big)\, d\zeta \quad \text{hence} \quad |g(z)| \le \int_0^1 |f'(tz) - f'(0)|\, |z|\, dt\,.$$

For the difference of derivatives, we obtain by the Cauchy Integral Formula

$$f'(z) - f'(0) = \frac{1}{2\pi i} \int_{\partial B_r(0)} f'(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta}\right) d\zeta = \frac{z}{2\pi i} \int_{\partial B_r(0)} \frac{f'(\zeta)}{\zeta(\zeta - z)}\, d\zeta\,.$$

The standard estimate then gives

$$|f'(z) - f'(0)| \le \frac{|z|}{r - |z|}\, |f'|_{\overline{B_r(0)}}\,.$$

We apply this to $|g(z)|$:

$$|g(z)| \le \int_0^1 |f'(tz) - f'(0)|\,|z|\,dt \le \int_0^1 \frac{t|z||f'|_{B_r(0)}}{r - t|z|}|z|\,dt$$
$$\le \frac{1}{2}\frac{|z|^2}{r-|z|}|f'|_{\overline{B_r(0)}} \le \frac{|z|^2}{r-|z|}|f'(0)|\,.$$

Let $|z| = \rho < r$. Then

$$|f(z)| \ge |f'(0)|\rho - |f(z) - f'(0)z| = |f'(0)|\rho - |g(z)| \ge \left(\rho - \frac{\rho^2}{r-\rho}\right)|f'(0)|\,.$$

The function $\rho \mapsto \big(\rho - \rho^2/(r-\rho)\big)|f'(0)|$ is maximal at $\rho = \rho^* := (1 - \sqrt{2}/2)r$, with value $R = (3 - 2\sqrt{2})r|f'(0)|$. So all the points on the circle of radius ρ^* are mapped outside the open disk of radius R centered at $0 = f(0)$. This implies that the image of f must contain $B_R(0)$, as we now show.

Let $U = f(B_{\rho^*}(0))$, and observe that ∂U is compact (it is a closed subset of the compact set $f(\overline{B_{\rho^*}(0)})$). So the distance $d = \operatorname{dist}(\partial U, 0)$ is realized by some $w \in \partial U$. We show that $d \ge R$. There is a sequence (z_n) in $B_{\rho^*}(0)$ such that $f(z_n) \to w$ and such that (z_n) converges to some $z \in \overline{B_{\rho^*}(0)}$. Since f is continuous, we have $f(z) = w$. Since f is open, z cannot be in $B_{\rho^*}(0)$, therefore we must have $|z| = \rho^*$. But from the argument above, we know that $d = |w| = |f(z)| \ge R$ in this case. □

We now proceed with the proof of Bloch's Theorem.

Proof. The idea is that we expect a large disk around $f(0)$ in the image if $|f'(0)|$ is large. We therefore try to modify f without changing its image in such a way that the derivative at the origin becomes large. A good way to modify f is to precompose it with an automorphism of the unit disk. Recall that

$$G := \operatorname{Aut}\big(B_1(0)\big) = \left\{z \mapsto \frac{\zeta z - w}{\bar{w}\zeta z - 1} : |\zeta| = 1, |w| < 1\right\}$$

(this can be proved using the Schwarz Lemma 6.14). All these maps extend to holomorphic functions on $B_{1/|w|}(0)$, so they are holomorphic on $\overline{B_1(0)}$. We consider

$$\mathcal{F} = \{f \circ j : j \in G\}\,.$$

For $j(z) = (\zeta z - w)/(\bar{w}\zeta z - 1)$, we have $|j'(0)| = 1 - |w|^2$, so

$$|(f \circ j)'(0)| = |f'(w)|(1 - |w|^2)\,.$$

By assumption, there is a point $q \in B_1(0)$ where $|f'(z)|(1 - |z|^2)$ attains its maximum $M > 0$. Then

$$F : z \longmapsto f\left(\frac{z - q}{\bar{q}z - 1}\right)$$

is in $\mathcal{F}$ and satisfies $|F'(0)| = M$. I now claim that

$$|F'(z)| \le \frac{M}{1 - |z|^2} \qquad \text{for all } z \in B_1(0).$$

To see this, observe first that $\mathcal{F} = \{F \circ j : j \in G\}$, since G is a group. Hence, for $w \in B_1(0)$, we have

$$M = \max_{h \in \mathcal{F}} |h'(0)| \ge \left|\frac{d}{dz}F\left(\frac{z - w}{\bar{w}z - 1}\right)\Big|_{z=0}\right| = |F'(w)|(1 - |w|^2)\,.$$

In particular, we have $|F'|_{B_{\sqrt{2}/2}(0)} \le 2M = 2|F'(0)|$. Lemma 16.3 then implies the theorem (take $r = \sqrt{2}/2$). $\square$

16.4. **Remark.** The constant $\frac{3}{2}\sqrt{2} - 2$ occurring in Bloch's Theorem 16.2 can be improved, see for example [R], Chapter 10 in Volume 2. According to this reference, the optimal value is strictly between 0.5 and 0.544.

By a suitable rescaling, we obtain the following consequence.

16.5. **Corollary.** *Let $U \subset \mathbb{C}$ be a domain, $f : U \to \mathbb{C}$ holomorphic. If $w \in U$ such that $f'(w) \neq 0$, then $f(U)$ contains open disks of every radius*

$$0 < r < \left(\frac{3}{2}\sqrt{2} - 2\right) |f'(w)| \operatorname{dist}(w, \partial U) .$$

In particular, if f is a non-constant entire function, then $f(\mathbb{C})$ contains arbitrarily large disks.

Proof. Without loss of generality, $w = 0$. Let $0 < s < \operatorname{dist}(0, \partial U)$, and consider $z \mapsto f(sz)$ on $\overline{B_1(0)}$. By Bloch's Theorem 16.2, $f(B_s(0))$ contains an open disk of radius $(\frac{3}{2}\sqrt{2}-2)s|f'(0)|$. (Note that $sf'(0)$ is the derivative of $z \mapsto f(sz)$ at 0.) $\square$

We will use the last statement in the corollary to prove Picard's Little Theorem. But first, we need a fairly elementary statement on entire functions.

16.6. **Lemma.** *Let $U \subset \mathbb{C}$ be a simply connected domain, and let $f : U \to \mathbb{C}$ be holomorphic such that $f(U)$ does not contain 0 nor 1. Then there is a holomorphic function $g : U \to \mathbb{C}$ such that $f = -\exp\bigl(\pi i \cosh(2g)\bigr)$.*

Proof. Since f has no zero in U and U is simply connected, there is a holomorphic function $h : U \to \mathbb{C}$ such that $f = \exp(2\pi i h)$. Since f does not take the value 1, neither h nor $h - 1$ can have zeros in U. Again since U is simply connected, there are holomorphic functions u and v on U such that $h = u^2$ and $h - 1 = v^2$. Then $1 = u^2 - v^2 = (u - v)(u + v)$, so $u - v$ never vanishes on U, hence there is a holomorphic function g on U such that $u - v = e^g$. Then $u + v = e^{-g}$, so $u = \cosh g$, and $\cosh(2g) = 2\cosh^2 g - 1 = 2h - 1$. It follows that

$$f = \exp(2\pi i h) = \exp\bigl(\pi i(1 + \cosh(2g))\bigr) = -\exp\bigl(\pi i \cosh(2g)\bigr) .$$

$\square$

16.7. **Corollary.** *Keep the notations of the previous lemma, and let*

$$A = \bigl\{\pm \log\bigl(\sqrt{m} + \sqrt{m-1}\bigr) + \tfrac{1}{2} n\pi i : m \in \mathbb{Z}_{>0}, n \in \mathbb{Z}\bigr\} .$$

Then $g(U) \cap A = \emptyset$. In particular, $g(U)$ does not contain open disks of radius 1.

Proof. Assume that there is $z \in U$ such that $g(z) \in A$. Then there are $m, n \in \mathbb{Z}$, $m > 0$, such that

$$e^{g(z)} = i^n\bigl(\sqrt{m} \pm \sqrt{m-1}\bigr) .$$

This would imply

$$e^{-g(z)} = i^{-n}\bigl(\sqrt{m} \mp \sqrt{m-1}\bigr) ,$$

hence

$$2\cosh(2g(z)) = (-1)^n\Bigl(\bigl(\sqrt{m} \pm \sqrt{m-1}\bigr)^2 + \bigl(\sqrt{m} \mp \sqrt{m-1}\bigr)^2\Bigr) = 2(-1)^n(2m - 1) .$$

But then

$$f(z) = -\exp\bigl(\pi i \cosh(2g)\bigr) = -\exp\bigl(\pi i (-1)^n (2m-1)\bigr) = 1\,,$$

a contradiction.

To prove the last statement, note that the points in A form the vertices of a mesh of rectangles in $\mathbb{C}$. The height of the rectangles is $\pi/2 < \sqrt{3}$, the width is

$$\log\bigl(\sqrt{m+1}+\sqrt{m}\bigr) - \log\bigl(\sqrt{m}+\sqrt{m-1}\bigr) = \log \frac{1+\sqrt{1+\frac{1}{m}}}{1+\sqrt{1-\frac{1}{m}}} \le \log(1+\sqrt{2}) < 1$$

(by the monotonicity of the logarithm). So the largest distance of a point in $\mathbb{C}$ from A is less than $\sqrt{(1/2)^2 + (\sqrt{3}/2)^2} = 1$, which means that every open disk of radius 1 in $\mathbb{C}$ must meet A. □

16.8. **Theorem (Picard's Little Theorem).** *Let f be an entire function such that $f(\mathbb{C})$ leaves out two distinct complex numbers a and b. Then f is constant.*

Proof. Without loss of generality, $a = 0$ and $b = 1$. (Consider $z \mapsto \frac{f(z)-a}{b-a}$.) Then by Lemma 16.6, there is an entire function g such that $f = -\exp\bigl(\pi i \cosh(2g)\bigr)$. By the corollary above, $g(\mathbb{C})$ does not contain an open disk of radius 1. By Cor. 16.5, this implies that $g' = 0$ everywhere, so g is constant, hence f is constant as well. □

Before we move on to Picard's (Great) Theorem, we want to prove a result that tells us that functions that are holomorphic on $\overline{B_1(0)}$ and leave out the values 0 and 1 can be bounded in a uniform way.

First, we need another consequence of Lemma 16.6

16.9. **Corollary.** *Let U and f be as in Lemma 16.6, with $0 \in U$, and let $r \ge 1$. Then there is $M(r) > 0$ such that if $r^{-1} \le |f(0)| \le r$, then we can choose g in such a way that $|g(0)| \le M(r)$.*

Proof. Assume that $r^{-1} \le |f(0)| \le r$. In the proof of Lemma 16.6, we can choose h so that $|\operatorname{Re} h(0)| \le \frac{1}{2}$ (note that h is only determined up to addition of an integer). We have $|f(z)| = e^{-2\pi \operatorname{Im} h(z)}$, so $\bigl|\log|f(0)|\bigr| = 2\pi|\operatorname{Im} h(0)|$, and

$$|h(0)| \le |\operatorname{Re} h(0)| + |\operatorname{Im} h(0)| \le \tfrac{1}{2} + \frac{\log r}{2\pi}\,.$$

Recall $u^2 = h$, $v^2 = h - 1$. We then have

$$|u(0) - v(0)| \le \sqrt{|h(0)|} + \sqrt{|h(0)-1|} \le P(r)$$

and

$$|u(0) - v(0)| = |u(0) + v(0)|^{-1} \ge \Bigl(\sqrt{|h(0)|} + \sqrt{|h(0)-1|}\Bigr)^{-1} \ge P(r)^{-1}$$

for a function $P(r) > 1$. Now recall that $e^g = u - v$, and choose g such that $|\operatorname{Im} g(0)| \le \pi$. Since $|u(z) - v(z)| = e^{\operatorname{Re} g(z)}$, we have

$$|g(0)| \le |\operatorname{Re} g(0)| + |\operatorname{Im} g(0)| \le \bigl|\log|u(0)-v(0)|\bigr| + \pi \le \log P(r) + \pi =: M(r)\,.$$

□

16.10. **Theorem (Schottky).** *Let $0 < \theta < 1$ and $r > 0$. Then there is a number $L(\theta, r) > 0$ such that for every holomorphic $f : \overline{B_1(0)} \to \mathbb{C} \setminus \{0, 1\}$, we have*

$$|f(z)| \le L(\theta, r) \qquad \textit{if } |z| \le \theta \textit{ and } |f(0)| \le r.$$

Proof. Let f satisfy the assumptions. Then by Lemma 16.6 and its corollaries, there is a function g, holomorphic on $\overline{B_1(0)}$, such that $f = -\exp\bigl(\pi i \cosh(2g)\bigr)$, and $g\bigl(B_1(0)\bigr)$ contains no open disks of radius 1.

Let $\beta = \frac{3}{2}\sqrt{2} - 2 > 0$ be the constant from Bloch's Theorem 16.2. Then by Cor. 16.5, we must have $|g'(w)| < \bigl(\beta(1 - |w|)\bigr)^{-1}$. Now let $z \in \overline{B_\theta(0)}$. Then

$$|g(z)| - |g(0)| \le |g(z) - g(0)| \le \int\limits_0^1 |g'(tz)z| \, dt \le \frac{\theta}{\beta(1-\theta)}\,,$$

so $|g(z)| \le |g(0)| + \theta/\bigl(\beta(1-\theta)\bigr)$.

We can assume that $r \ge 2$. If $|f(0)| \ge r^{-1}$, then we can assume that $|g(0)| \le M(r)$, where $M(r)$ is as in Cor. 16.9. Then

$$|f(z)| \le \exp\bigl(\pi \cosh|2g(z)|\bigr) \le \exp\left(\pi \cosh\left(2M(r) + 2\frac{\theta}{\beta(1-\theta)}\right)\right) =: L_1(\theta, r)\,.$$

If $|f(0)| \le r^{-1} \le \frac{1}{2}$, then $\frac{1}{2} \le |1 - f(0)| \le 2$, and we can apply the previous reasoning to $1 - f$, which gives

$$|f(z)| \le 1 + |1 - f(z)| \le 1 + L_1(\theta, 2)\,.$$

If we set

$$L(\theta, r) = \max\{L_1(\theta, \max\{2, r\}), 1 + L_1(\theta, 2)\}\,,$$

we have a function with the required property. □

This result is stronger than Picard's Little Theorem 16.8: it gives an 'effective' version of it.

16.11. **Theorem (Landau).** *There is a function $R : \mathbb{C} \setminus \{0, 1\} \to \mathbb{R}_+$ such that there is no holomorphic function f on $\overline{B_{R(a)}(0)}$ with $f(0) = a$, $f'(0) = 1$, and such that f leaves out the values 0 and 1.*

Proof. Define $R(a) = 3L(\frac{1}{2}, |a|)$, where L is as in Schottky's Theorem 16.10. If $f(z) = a + z + \ldots$ leaves out the values 0 and 1, then the same is true for $g(z) = f(Rz)$, where $R = R(a)$. By Schottky's Theorem, we would have

$$\max\bigl\{|g(z)| : |z| = \tfrac{1}{2}\bigr\} \le \tfrac{1}{3}R\,.$$

On the other hand, Cuachy's inequalities for the Taylor coefficients of g (see Cor. 5.18) imply that $R \le 2\max\bigl\{|g(z)| : |z| = \frac{1}{2}\bigr\}$, a contradiction. □

Schottky's Theorem leads to considerable improvements in Montel's and Vitali's Theorems.

16.12. **Corollary.** *Let $U \subset \mathbb{C}$ be a domain, $w \in U$, $r > 0$, and $\mathcal{F}$ a family of holomorphic functions $U \to \mathbb{C} \setminus \{0,1\}$ such that $|f(w)| \leq r$ for all $f \in \mathcal{F}$. Then there is a neighborhood $V \subset U$ of w such that $\mathcal{F}$ is bounded on V.*

Proof. Let $t > 0$ such that $\overline{B_{2t}(w)} \subset U$. By Schottky's Theorem 16.10, applied to $z \mapsto f(w + 2tz)$, $|z| \leq 1$, $f \in \mathcal{F}$, we have $|f(z)| \leq L(\frac{1}{2}, r)$ for all $f \in \mathcal{F}$ and $z \in V = B_t(w)$. $\square$

16.13. **Theorem (Strong Montel).** *Let $U \subset \mathbb{C}$ be a domain, and let $a, b, c \in \hat{\mathbb{C}}$ be three distinct points. Let $\mathcal{F}$ be the family of all holomorphic maps from U into $\hat{\mathbb{C}} \setminus \{a, b, c\}$. Then $\mathcal{F}$ is normal (with respect to the spherical metric).*

Proof. We may assume that $\{a, b, c\} = \{0, 1, \infty\}$, so $\mathcal{F}$ is the family of all holomorphic functions on U leaving out the values 0 and 1. Let $w \in U$, and let $\mathcal{F}_1 = \{f \in \mathcal{F} : |f(w)| \leq 1\}$. I claim that $\mathcal{F}_1$ is locally bounded on U. To see this, let

$$A = \{z \in U : \mathcal{F}_1 \text{ is bounded on a neighborhood of } z\} \subset U\,;$$

then A is clearly open. By Cor. 16.12, $w \in A$, so A is non-empty. Now let $z \in U \setminus A$. By Cor. 16.12 again, the set $\{f(z) : f \in \mathcal{F}_1\}$ is unbounded, so there is a sequence (f_n) in $\mathcal{F}_1$ such that $f_n(z) \to \infty$. Consider $g_n = 1/f_n$; we have $g_n \in \mathcal{F}$ and $g_n(z) \to 0$, so $\{g_n : n \in \mathbb{N}\}$ is (again by Cor. 16.12) bounded on a neighborhood of z. By Montel's Theorem 13.3, there is a subsequence $(g_{n_k})_k$ that converges uniformly in a disk V around z to a holomorphic function g. Since none of the g_n has a root, but $g(z) = 0$, we must have $g = 0$ by Thm. 10.7. But then $f_{n_k}(\zeta) \to \infty$ as $k \to \infty$ for all points $\zeta \in V$, so $V \subset U \setminus A$, and $U \setminus A$ is open as well. Since U is connected, it follows that $A = U$, and $\mathcal{F}_1$ is locally bounded.

Now let (f_n) be a sequence in $\mathcal{F}$. If this sequence has a subsequence that is in $\mathcal{F}_1$, then there are locally bounded subsequences, and the claim follows from Montel's Theorem 13.3. Otherwise, there are only finitely many terms f_n in $\mathcal{F}_1$, hence all but finitely many f_n satisfy $1/f_n \in \mathcal{F}_1$, so there is a subsequence f_{n_k} such that $1/f_{n_k}$ converges compactly in $\hat{\mathbb{C}}$ as $k \to \infty$. But then f_{n_k} converges compactly as well, since $z \mapsto 1/z$ is an automorphism of $\hat{\mathbb{C}}$. $\square$

16.14. **Theorem (Carathéodory-Landau).** *Let $U \subset \mathbb{C}$ be a domain, let $a, b, c \in \hat{\mathbb{C}}$ be distinct, and let $f_n : U \to \hat{\mathbb{C}} \setminus \{a, b, c\}$ be a sequence of holomorphic functions such that $\lim f_n(z) \in \hat{\mathbb{C}}$ exists for $z \in A$, where $A \subset U$ has at least one accumulation point in U. Then (f_n) converges compactly on U (with respect to the spherical metric).*

Proof. This follows from Thm. 16.13 in the same way as Vitali's Theorem 13.5 followed from Montel's Theorem 13.3. $\square$

Finally, we obtain Picard's Theorem.

16.15. **Theorem (Picard).** *Let $a \in \mathbb{C}$ be an essential isolated singularity of f. Then with at most one exception, f assumes every value in $\mathbb{C}$ infinitely often in any punctured neighborhood of a, on which f is defined.*

Proof. Assume that there are at least two exceptions; as usual, we can assume that they are 0 and 1. It suffices to show that if f leaves out these two values on some punctured neighborhood of a, either f or $1/f$ must be bounded near a. For this, we can in turn assume that $a = 0$ and f is defined on $B_1(0) \setminus \{0\}$.

The family $\{f_n = z \mapsto f(z/n) : n \geq 1\}$ of holomorphic functions on the punctured unit disk leaves out the values 0 and 1, so by the proof of the 'Strong Montel' Theorem 16.13, there is a subsequence (f_{n_k}) such that either the f_{n_k} or the $1/f_{n_k}$ are bounded on the circle $|z| = \frac{1}{2}$. We can assume that $n_1 = 1$. In the first case, let M be a bound for $|f(\frac{z}{n_k})| = |f_{n_k}(z)|$, for all $|z| = \frac{1}{2}$, and all $k \geq 1$. Then $|f| \leq M$ on every circle $|z| = \frac{1}{2n_k}$, hence by the Maximum Principle 6.13, $|f| \leq M$ on every annulus $\frac{1}{2n_k} \leq |z| \leq \frac{1}{2}$, so that f is bounded on a neighborhood of $a = 0$. In the other case, we see in the same way that $1/f$ is bounded on a neighborhood of $a = 0$. □

References

[A] L.V. Ahlfors: *Complex analysis,* Third edition. McGraw-Hill, 1979.

[C] J.B. Conway: *Functions of one complex variable, I,* Second edition. Springer Graduate Texts in Mathematics **11**, 1978.

[J] K. Jänich: *Einführung in die Funktionentheorie.* (German) Springer Hochschultext, 1980.

[R] R. Remmert: *Funktionentheorie I, II.* (German) Springer Grundwissen Mathematik vols. **5** and **6**, 1984, 1991.