Construction of Two-Weight Codes

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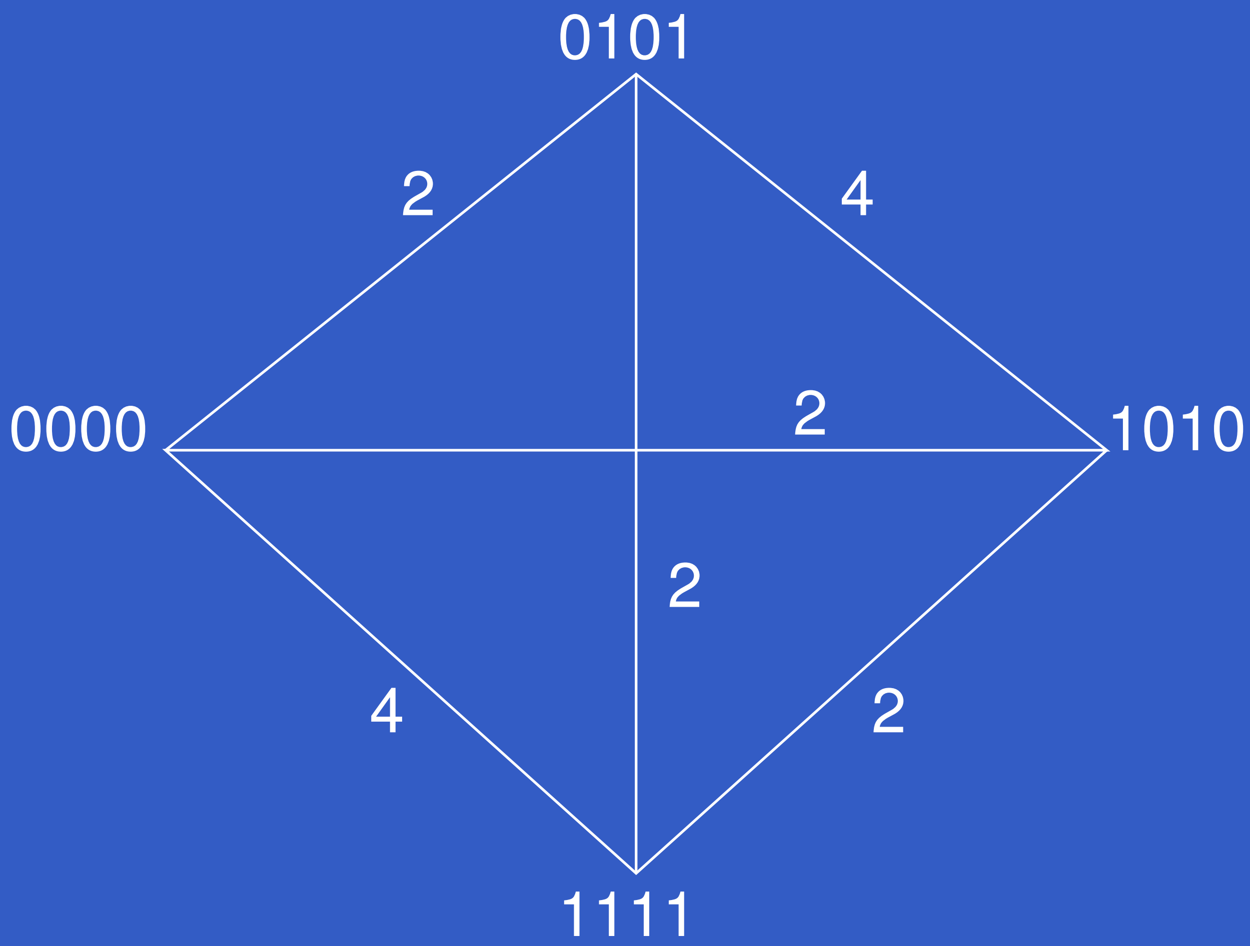
Coding Theory



Coding Theory

Hamming distance $d_H(x, y)$=number of places with different letters in two codewords x and y.

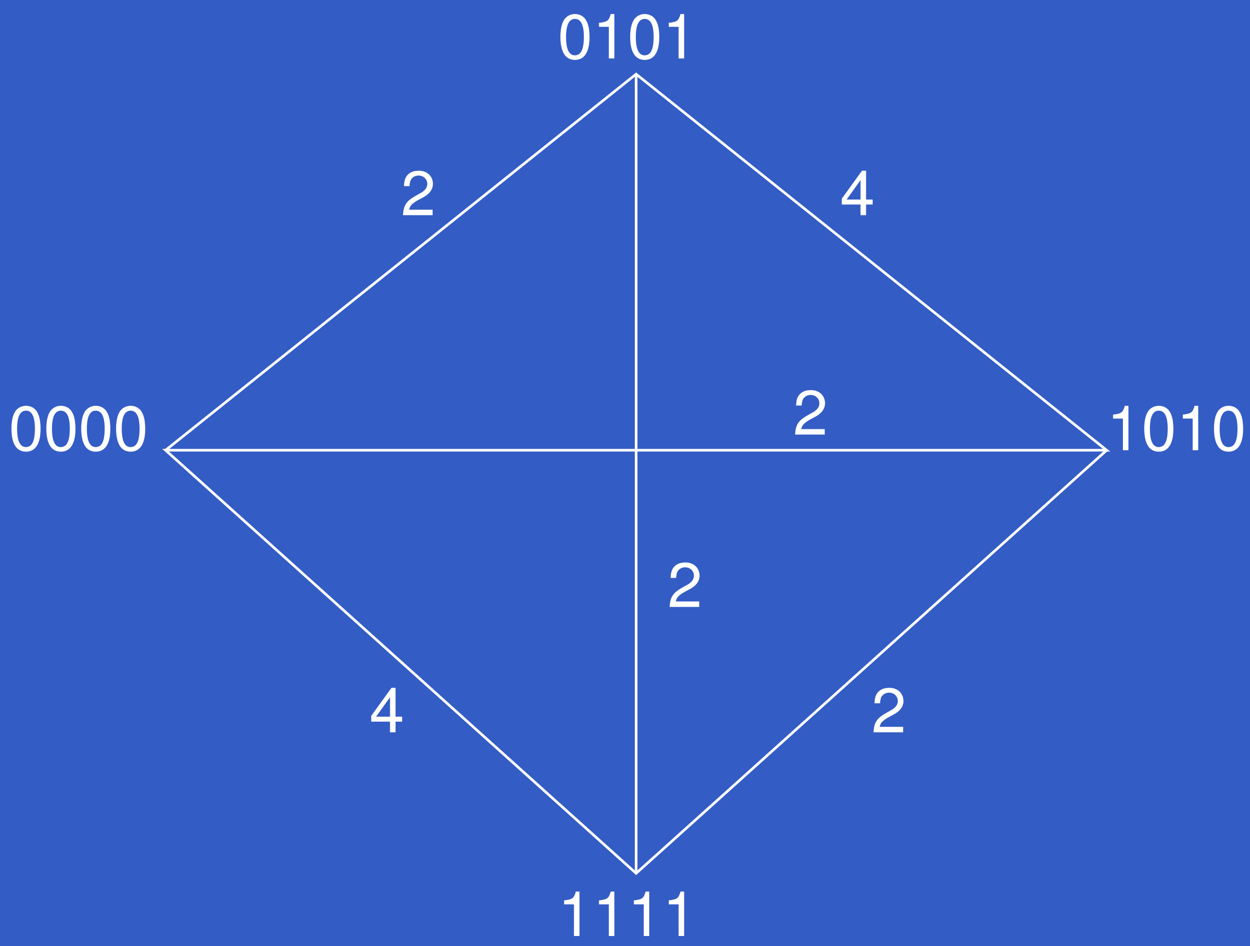
Minimum distance = minimum of $d_H(x, y)$ for all pairs of codewords.

Error correcting capability is measured by the minimum distance.

Linear Code

A **linear** $[n, k; q]$ **code** C is a $k-$dimensional subspace of the vectorspace $GF(q)^n$.

Linear Code



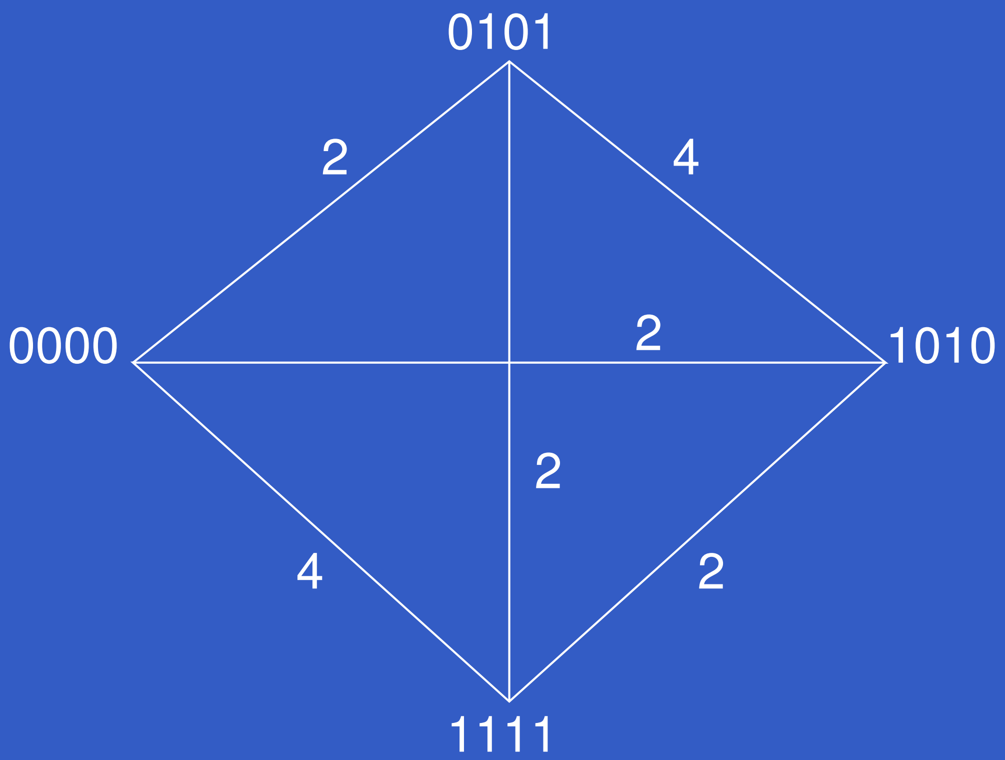
Linear Code

A **linear** $[n, k; q]$ **code** C is a $k-$dimensional subspace of the vectorspace $GF(q)^n$.

The **generator matrix** Γ of a linear $[n, k; q]$ code C is a $k \times n$ matrix where each row is a basis element of the code C.

$$C = \{v\Gamma : v \in GF(q)^k\}$$

Linear Code



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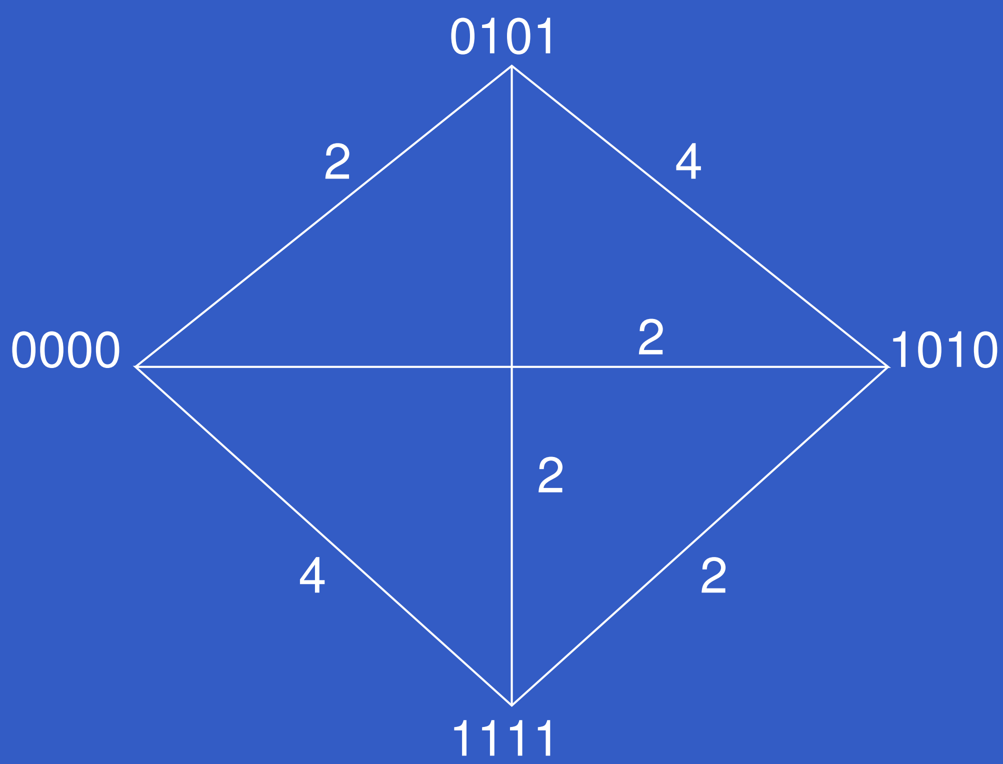

$$\Gamma = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Minimum Distance

The minimum distance of a linear code is the minimum number of nonzero entries (=**weight**) of all nonzero codewords.

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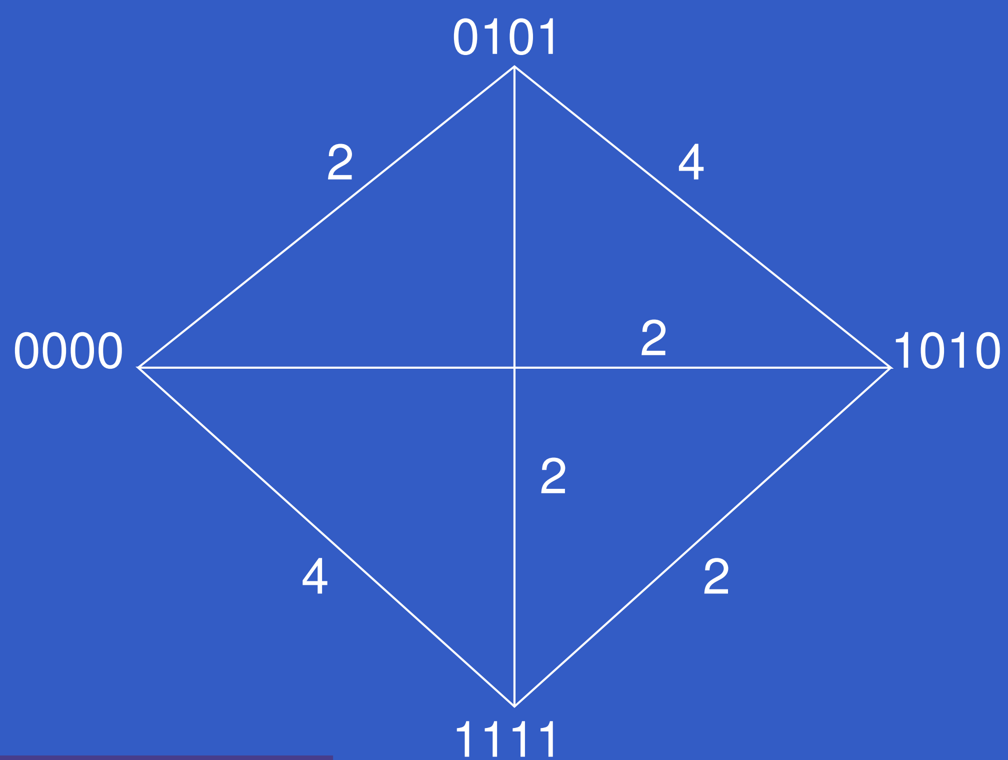

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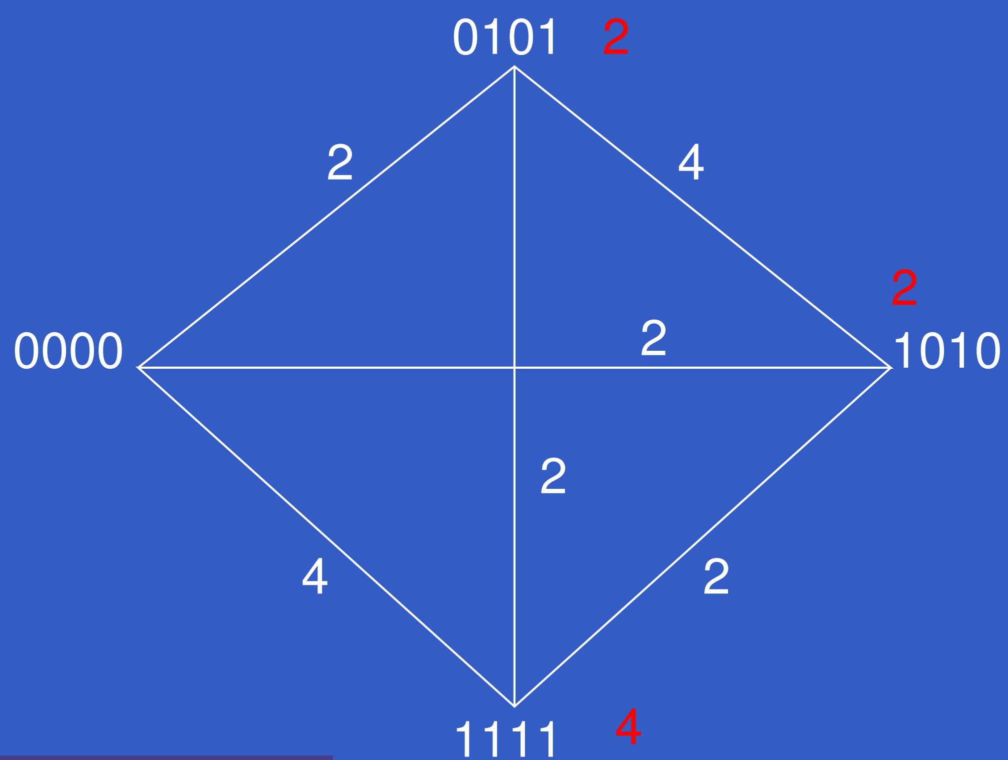

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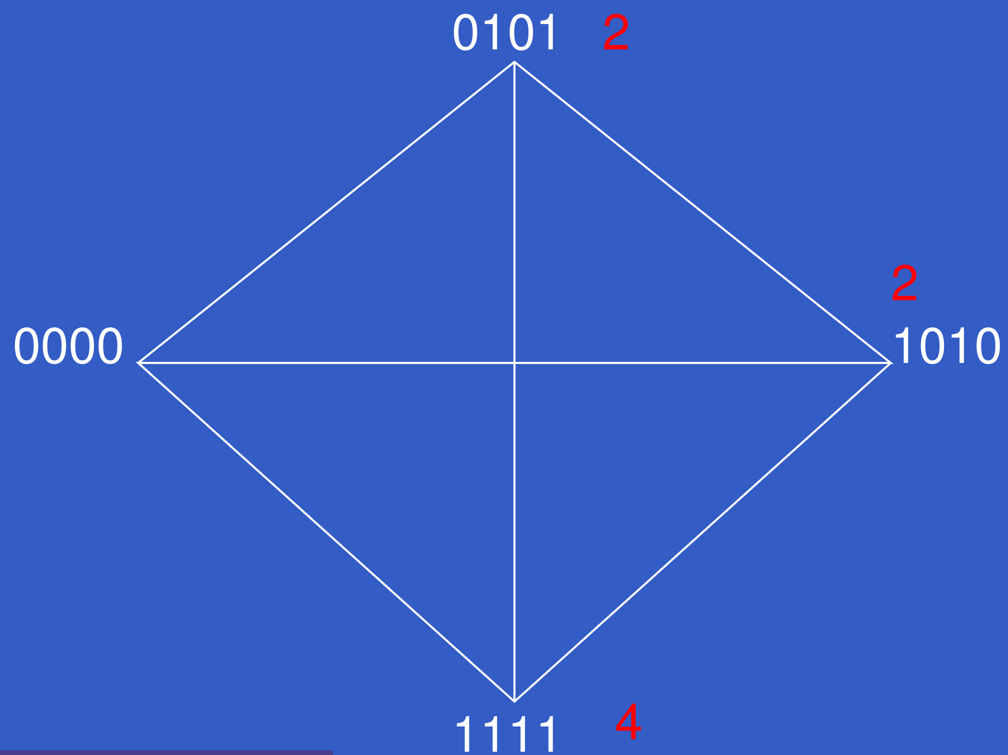

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Weight Enumerator

Weight enumerator $A_C(z) := \sum A_i z^i$ where A_i is the number of codewords in C of weight i.

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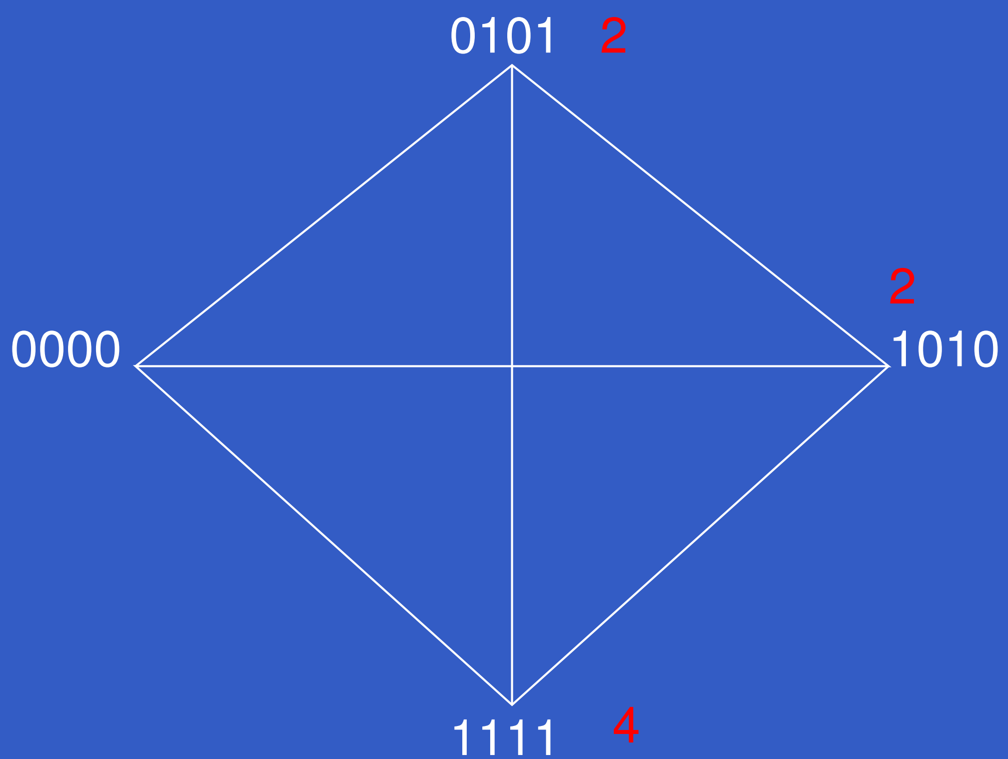

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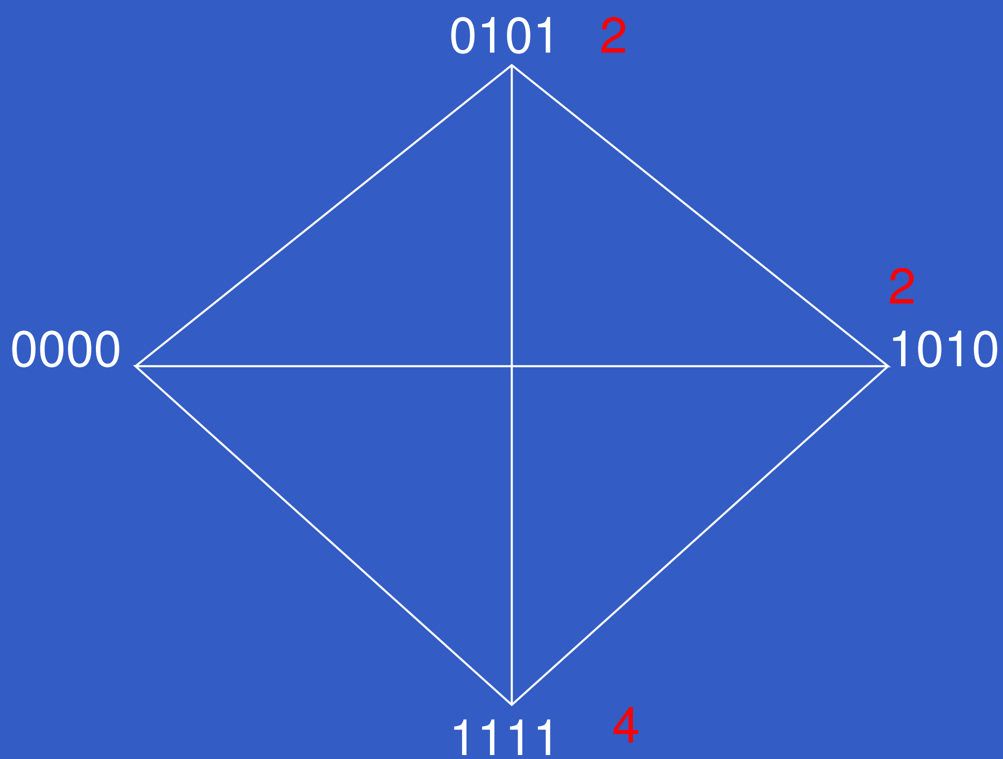

$$A_C = z^0 + 2z^2 + z^4$$

Two-Weight Code

This is a (linear) code with only two different nonzero weights w_1 and $w_2 (w_1 < w_2)$.

$$
\begin{matrix}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1
\end{matrix}
$$

Two-Weight Code

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$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \quad \Gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

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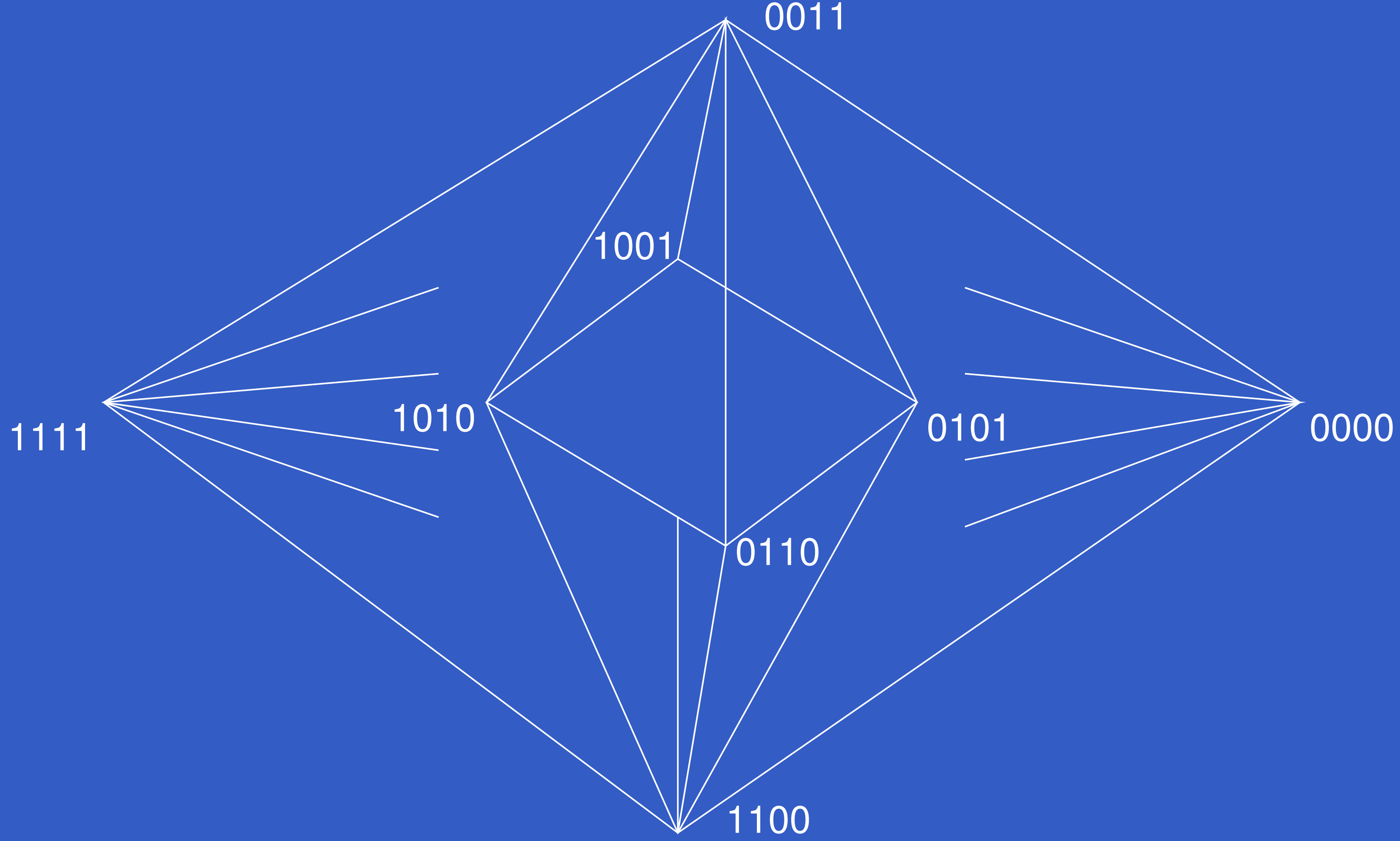
Two-Weight Code

Visualization of a two-weight code C by a graph G_C



Graph G_C of a Two-Weight Code

Given a code C with the two nonzero weights w_1 and w_2

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vertices = codewords

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Given a code C with the two nonzero weights w_1 and w_2

vertices = codewords

edge between x and y if $d_H(x, y) = w_1$

Properties of G_C

G_C is a **regular** graph.

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G_C is a **regular** graph.

G_C is **strongly regular** [DELSARTE], i.e. the number of common neighbors of a pair x, y of vertices depends only on the fact whether x and y are adjacent or not.

Strongly Regular Graphs

A strongly regular graph is (partially) described by four parameters (N, K, λ, μ)

N= number of vertices

K= degree

λ = number of common neighbors of adjacent vertices

μ = number of common neighbors of non-adjacent vertices

Construction

To construct a two-weight $[n, k; q]$ code we construct a corresponding generator matrix Γ.

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The codewords of a two-weight code have $n - w_1$ or $n - w_2$ zeros.

We have to control the number of zeros in the codewords.

Construction

A codeword c is given by a product:

$$v\Gamma = c. \qquad\qquad (v \in GF(q)^k)$$

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We build a matrix M whose columns are labeled by the possible columns γ of the generator matrix. Rows are labeled by the nonzero $v \in GF(q)^k$ which give after multiplication with the generator matrix the codewords of the two-weight code.

Weight Matrix

$$M = \boxed{}$$

Weight Matrix

$$\gamma \in GF(q)^k$$

$$\downarrow$$

$$M = \boxed{M_{v,\gamma}} \leftarrow v \in GF(q)^k$$

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$$M = \boxed{\quad M_{v,\gamma} \quad} \leftarrow v \in GF(q)^k$$

$$M_{v,\gamma} = \begin{cases} 1 & v\gamma = 0 \\ 0 & v\gamma \neq 0 \end{cases}$$

Diophantine System of Equations

Now a two-weight code corresponds to a $0/1$ solution $x = (x_1, \ldots, x_{q^k-1})$ of the system

$$
\begin{array}{lccc}
(1) & Mx & = & \begin{pmatrix} n - w_1 \text{ or } n - w_2 \\ \vdots \\ n - w_1 \text{ or } n - w_2 \end{pmatrix} \\
 & & & \\
(2) & \sum x_i & = & n
\end{array}
$$

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Diophantine System of Equations

$$\left[\begin{array}{c|ccccc} & w_1 - w_2 & 0 & \dots & 0 & 0 \\ & 0 & \ddots & 0 & & 0 \\ M & \vdots & 0 & w_1 - w_2 & 0 & \vdots \\ & 0 & & 0 & \ddots & 0 \\ & 0 & 0 & \dots & 0 & w_1 - w_2 \\ \hline 1 \dots 1 & 0 & \dots & 0 & \dots & 0 \end{array}\right] x = \left[\begin{array}{c} n - w_1 \\ \vdots \\ \\ \vdots \\ n - w_1 \\ \hline n \end{array}\right]$$

Diophantine System of Equations

$$
\left[\begin{array}{c|ccccc}
 & w_1 - w_2 & 0 & \dots & 0 & 0 \\
 & 0 & \ddots & 0 & & 0 \\
M & \vdots & 0 & w_1 - w_2 & 0 & \vdots \\
 & 0 & & 0 & \ddots & 0 \\
 & 0 & 0 & \dots & 0 & w_1 - w_2 \\
\hline
1 \dots 1 & 0 & \dots & 0 & \dots & 0
\end{array}\right] x = \left[\begin{array}{c}
n - w_1 \\
\vdots \\
\\
\vdots \\
n - w_1 \\
\hline
n
\end{array}\right]
$$

To solve this system we use an LLL-variant of A. Wassermann.

Diophantine System of Equations

We are interested in a $0/1$ solution $x = (x_1, \ldots, x_{q^k-1}, \ldots, x_{2(q^k-1)})$ of the system.

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The first half $x = (x_1, \ldots, x_{q^k-1})$ of a solution corresponds via selection of columns of the generator matrix to an $[n, k; q]$ two-weight code with weights w_1 and w_2.

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The first half $x = (x_1, \ldots, x_{q^k-1})$ of a solution corresponds via selection of columns of the generator matrix to an $[n, k; q]$ two-weight code with weights w_1 and w_2.

The second half $x = (x_{q^k}, \ldots, x_{2(q^k-1)})$ contains the information on the weight enumerator.

Projective Geometry

As we are computing scalar products, the 0/nonzero property is invariant under scalar multiplication, so we can label rows and columns by $1-$dimensional subspaces of $GF(q)^k$.

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Projective Geometry

As we are computing scalar products, the 0/nonzero property is invariant under scalar multiplication, so we can label rows and columns by $1-$dimensional subspaces of $GF(q)^k$.

M is after this reduction the incidence matrix between the $1-$dimensional subspaces and the $(k-1)-$ dimensional subspaces of $GF(q)^k$.

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3 Different Languages

We can study the same object in 3 different settings:

- Two-Weight Codes
- Strongly Regular Graphs
- Point-Sets in the Projective Geometry

Automorphisms

We further reduce the size of the system by prescribing a group of automorphisms, this method corresponds to choosing complete orbits of subgroups of $PGL(k,q)$ on the $1-$dimensional subspaces as possible columns of the generator matrix.

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This further reduces the number of columns, in our system of equations, as the dimension is now the number of orbits.

Reduction

The defining property of the incidence matrix

$$M_{U,V} = 1 \iff U \leq V$$

is invariant under the automorphisms.

Reduction

The defining property of the incidence matrix

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This also reduces the number of rows in the same way, the height is also the number of orbits.

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Example

We computed a new $[738, 8; 3]$ two-weight code with nonzero weights 486 and 513.

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$$6560$$

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$$q^k - 1 \qquad \frac{q^k-1}{q-1}$$

$$6560 \quad \longrightarrow \quad 3280$$

Example

We computed a new $[738, 8; 3]$ two-weight code with nonzero weights 486 and 513.

$$q^k - 1 \qquad \frac{q^k-1}{q-1} \qquad \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & 2 & 2 & 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 2 & 0 & 2 & 2 & 0 \\ 1 & 2 & 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 & 2 & 2 & 0 \end{pmatrix}$$

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$6560 \quad \longrightarrow \quad 3280 \quad \longrightarrow \quad 40$ orbits

Searching for Groups

We use different subgroups of $PGL(k, q)$.

- random cyclic generator (like above example)
- Permutation groups
- Blockdiagonal
- Monomial

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Limits on orbit sizes, number of orbits, ...

Results

Using this method we computed several new two-weight codes.

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Among these there are also distance-optimal codes.

Results

Some new two-weight codes

two-weight code					strongly regular graph			
n	k	q	w_1	w_2	N	K	λ	μ
140	6	3	90	99	729	280	103	110
198*	10	2	96	112	1024	198	22	42
...								

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Last Page

Thank you very much for your attention.

- A. Kohnert: Construction of Two-Weight Codes, in preparation
- M. Braun, A. Kohnert, A. Wassermann: Optimal Linear Codes From Matrix Groups, IEEE Information Theory, 2005

Last Page

Thank you very much for your attention.

- list of new codes including generator matrix and weight enumerator:
  http://linearcodes.uni-bayreuth.de
- A. E. Brouwer has a list (not online) of known parameters:
  http://www.win.tue.nl/~aeb/