# APPLICATIONS OF THE CLASSICAL UMBRAL CALCULUS

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*Dedicated to the memory of Gian-Carlo Rota*

Abstract. We describe applications of the classical umbral calculus to bilinear generating functions for polynomial sequences, identities for Bernoulli and related numbers, and Kummer congruences.

## 1. Introduction

In the nineteenth century, Blissard developed a notation for manipulating sums involving binomial coefficients by expanding polynomials and then replacing exponents with subscripts. For example, the expression $(a+1)^n$ would represent the sum $\sum_{i=0}^{n}\binom{n}{i}a_i$. Blissard's notation has been known variously as Lucas's method, the symbolic method (or symbolic notation), and the umbral calculus. We shall use Rota and Taylor's term "classical umbral calculus" [37] to distinguish it from the more elaborate mathematical edifice that the term "umbral calculus" has come to encompass [32, 33, 35].

The goal of this article is to show, by numerous examples, how the classical umbral calculus can be used to prove interesting formulas not as easily proved by other methods. Our applications are in three general areas: bilinear generating functions, identities for Bernoulli numbers and their relatives, and congruences for sequences such as Euler and Bell numbers.

The classical umbral calculus is intimately connected with exponential generating functions; thus $a^n = a_n$ is equivalent to

$$e^{ax} = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!},$$

and multiplication of exponential generating functions may be expressed compactly in umbral notation:

$$\left(\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} b_n \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$$

is equivalent to $(a+b)^n = c^n$.

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When I first encountered umbral notation it seemed to me that this was all there was to it; it was simply a notation for dealing with exponential generating functions, or to put it bluntly, it was a method for avoiding the use of exponential generating functions when they really ought to be used. The point of this paper is that my first impression was wrong: none of the results proved here (with the exception of Theorem 7.1, and perhaps a few other results in section 7) can be easily proved by straightforward manipulation of exponential generating functions. The sequences that we consider here are defined by exponential generating functions, and their most fundamental properties can be proved in a straightforward way using these exponential generating functions. What is surprising is that these sequences satisfy additional relations whose proofs require other methods. The classical umbral calculus is a powerful but specialized tool that can be used to prove these more esoteric formulas. The derangement numbers, for example, have the well-known exponential generating function $\sum_{n=0}^{\infty} D_n x^n/n! = e^{-x}/(1-x)$ from which their basic properties can be derived; umbral calculus gives us the more interesting but considerably more recondite formula $\sum_{n=0}^{\infty} D_n^2 x^n/n! = e^x \sum_{k=0}^{\infty} k!\, x^k/(1+x)^{2k+2}$.

We begin in the next section with a description of the classical umbral calculus, following Rota [34] and Rota and Taylor [36, 37], and point out some of the minor ways in which we differ from their approach. Next, in sections 3 through 5, we consider bilinear and related generating functions for Charlier and Hermite polynomials, and some variations. In section 6 we derive a bilinear generating function for the Rogers-Szegő polynomials, which are related to $q$-Hermite polynomials. In section 7 we apply the umbral calculus to identities for Bernoulli and related numbers. Sections 8 through 10 deal with Kummer congruences and with analogous congruences for Bell numbers.

The next section contains a formal description of the classical umbral calculus as used in this paper. The reader who is not interested in these technicalities may wish to go directly to section 3.

## 2. The classical umbral calculus

Most users of Blissard's symbolic notation have viewed it as simply a notational convenience, requiring no formal justification. Thus Guinand [23], in explaining the interpretation of umbral symbols, writes: "In general, any step in manipulation is valid if and only if it remains valid when interpreted in non-umbral form." However, in 1940 E. T. Bell [5] attempted to give an axiomatic foundation to the umbral calculus. To the modern reader, Bell's approach seems ill-conceived, if not completely incomprehensible. A much more successful explanation was given by G.-C. Rota in 1964 [34]: When we interpret $(a+1)^n$ as $\sum_{i=0}^{n} \binom{n}{i} a_i$, we are applying the linear functional on the algebra of polynomials in $a$ that takes $a^i$ to $a_i$. In retrospect, Rota's idea seems almost obvious, but we must remember that in Bell's day the concept of a linear functional was not the familiar notion that it is in ours. The seemingly mysterious "umbral variable" $a$ is just an ordinary variable; it is in the invisible, but otherwise unremarkable, linear functional that the meaning of the umbral calculus resides. The "feeling of witchcraft" that Rota and Taylor [37] observe hovering about the umbral calculus comes from the attribution to umbrae of properties that really belong to these linear functionals. As in stage illusion, misdirection is essential to the magic.

Rota and Taylor's recent works [36, 37] expanded on Rota's original insight and introduced new concepts that help to resolve some of the ambiguities that may arise in applications of the traditional notation. However, I shall use the traditional notation in this paper. What follows is a short formal description of the classical umbral calculus as used here, based on Rota and Taylor's formulation, but with some modifications.

In the simplest applications of the classical umbral calculus, we work in the ring of polynomials in one variable, e.g., $R[a]$, where $R$ is a ring of "scalars" ($R$ is often a ring of polynomials or formal power series containing the rationals), and we have a linear functional $\mathrm{eval} : R[a] \to R$. (This notation was introduced by Rota and Taylor [36].) The variable $a$ is called an umbral variable or *umbra*. There is nothing special about it other than the fact that the linear functional eval is defined on $R[a]$. We will often use the same letter for the umbra and the sequence; thus we would write $a_n$ for $\mathrm{eval}(a^n)$. It is traditional, and convenient, to omit eval and to write $a^n = a_n$ instead of $\mathrm{eval}(a^n) = a_n$. However when following this convention, we must make clear where eval is to be applied. The rule that we shall follow in this paper is that eval should be applied to any term in an equation that contains a variable that has been declared to be umbral. It should be emphasized that this is a syntactic, not mathematical rule, so the formula $a^n = n$ is to be interpreted as $\mathrm{eval}(a^n) = n$ for all $n$, even though for $n = 0$, $a$ does not "appear" on the left side. One important difference between our approach and that of Rota and Taylor [36, 37] is that they require that $\mathrm{eval}(1) = 1$, but we do not, and in sections 7 and 9 we shall see several examples where $\mathrm{eval}(1) = 0$. This involves some notational subtleties discussed below; nevertheless, there is no reason why a linear functional on polynomials cannot take 1 to 0, and there are are interesting applications where this happens.

We shall often have occasion to deal with several umbrae together. It should be pointed out that although we use the symbol eval for whatever linear functional is under discussion, there are really many different such functionals. When we write $a^n = a_n$ and $b^n = b_n$ we are really talking about two different linear functionals, $\mathrm{eval}_1 : R[a] \to R$ and $\mathrm{eval}_2 : R[b] \to R$, where $\mathrm{eval}_1(a^n) = a_n$ and $\mathrm{eval}_2(b^n) = b_n$. The meaning of $\mathrm{eval}(a^m b^n)$ might be determined by a completely different linear functional on $R[a, b]$, but traditionally one takes the linear functional $\mathrm{eval}_3$ defined by $\mathrm{eval}_3(a^m b^n) = \mathrm{eval}_1(a^m)\,\mathrm{eval}_2(b^n)$. In this case, we say that the umbrae $a$ and $b$ are *independent* (even though we are really dealing with a property of the linear functional $\mathrm{eval}_3$ rather than a property of the variables $a$ and $b$). In fact, applications of umbrae that are not independent in this sense are uncommon and do not seem to be have been considered before, and we shall assume that our umbrae are independent except where we explicitly state otherwise. Nevertheless we give an example in section 5 of an application of umbrae that are not independent.

Eschewing the requirement that $\mathrm{eval}(1) = 1$ entails an additional interpretative issue that must be mentioned. We cannot assume that there is a "universal" evaluation functional that applies to every term in a formula; instead we may need a different functional for each term, corresponding to the variables that appear in that term. In section 9, for example, we have the formula

$$F^n = 2A^n - (4B + C)^n,$$

involving the umbrae $F$, $A$, $B$, and $C$, which must be interpreted as

$$\mathrm{eval}_1(F^n) = \mathrm{eval}_2(2A^n) - \mathrm{eval}_3\big((4B + C)^n\big),$$

where $\mathrm{eval}_1$ is defined on $\mathbf{Q}[F]$, $\mathrm{eval}_2$ is defined on $\mathbf{Q}[A]$, and $\mathrm{eval}_3$ is defined on $\mathbf{Q}[B,C]$. Although the rule may seem unnatural when stated this way, in practice the interpretation is exactly what one would expect.

We will often find it useful to work with power series, rather than polynomials, in our umbrae. However, if $f(u)$ is an arbitrary formal power series in $u$ and $a$ is an umbra then $\mathrm{eval}(f(a))$ does not make sense. Let us suppose that $R$ is a ring of formal power series in variables $x, y, z, \ldots$. Then we call a formal power series $f(u) \in R[[u]]$ *admissible* if for every monomial $x^i y^j z^k \cdots$ in $R$, the coefficient of $x^i y^j z^k \cdots$ in $f(u)$ is a polynomial in $u$. Then if $f(u) = \sum_i f_i u^i$ is admissible, we define $\mathrm{eval}(f(a))$ to be $\sum_i f_i \,\mathrm{eval}(a^i)$; admissibility of $f$ ensures that this sum is well defined as an element of $R$. More generally, we may define admissibility similarly for a formal power series in any finite set of variables with coefficients involving other variables.

## 3. Charlier polynomials

In the next three sections we apply the classical umbral calculus to find bilinear generating functions. More specifically, we find explicit expressions for generating functions of the form $\sum_n a_n b_n x^n/n!$, where there are simple expressions for the generating functions $\sum_n a_n x^n/n!$ and $\sum_n b_n x^n/n!$. Although it is not obvious *a priori* that such explicit expressions exist, they do, and they have important applications in the theory of orthogonal polynomials (see, e.g., Askey [3]). The method that we use can be translated into a traditional analytic computation, since in all cases that we consider in these three sections, eval can be represented by a definite integral (though in some cases the radius of convergence of the series is 0). For example, in this section we consider the umbra $A$ evaluated by $\mathrm{eval}(A^n) = \alpha(\alpha+1)\cdots(\alpha+n-1)$. We could define eval analytically by

$$\mathrm{eval}\big(f(A)\big) = \frac{1}{\Gamma(\alpha)} \int_0^\infty f(x) x^{\alpha-1} e^{-x}\, dx$$

and do all our calculations with integrals. In fact this idea has been used, in a significantly more sophisticated setting, by Ismail and Stanton [25, 26, 27] to obtain bilinear generating functions much more complicated than those we deal with here.

The rising factorial $(\alpha)_n$ is defined to be $\alpha(\alpha+1)\cdots(\alpha+n-1)$. The Charlier polynomials $c_n(x;a)$ are defined by

$$c_n(x;a) = \sum_{k=0}^{n} \binom{n}{k} (-x)_k\, a^{-k}$$

(see, for example, Askey [3, p. 14]), but it is more convenient to work with differently normalized versions of these polynomials, which we define as

$$C_n(u,\alpha) = u^n c_n(-\alpha; u) = \sum_{i=0}^{n} \binom{n}{i} (\alpha)_i u^{n-i}.$$

Let us define the umbra $A$ by $A^n = (\alpha)_n$. Then

$$C_n(u,\alpha) = (A+u)^n. \tag{3.1}$$

Now

$$e^{Ax} = \sum_{n=0}^{\infty} A^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\alpha)_n \frac{x^n}{n!} = (1-x)^{-\alpha}, \tag{3.2}$$

by the binomial theorem. So

$$\sum_{n=0}^{\infty} C_n(u,\alpha) \frac{x^n}{n!} = e^{(A+u)x} = e^{ux} e^{Ax} = \frac{e^{ux}}{(1-x)^{\alpha}}. \tag{3.3}$$

Our goal in this section is to prove the bilinear generating function for the Charlier polynomials,

$$\sum_{n=0}^{\infty} C_n(u,\alpha) C_n(v,\beta) \frac{x^n}{n!} = e^{uvx} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(1-vx)^{k+\alpha}} \frac{(\beta)_k}{(1-ux)^{k+\beta}} \frac{x^k}{k!}.$$

To do this we first prove some properties of the umbra $A$.

**Lemma 3.1.** *For any admissible formal power series $f$,*

$$e^{Ay} f(A) = \frac{1}{(1-y)^{\alpha}} f\left(\frac{A}{1-y}\right).$$

*Proof.* First we prove the lemma for the case $f(z) = e^{zw}$. We have

$$\begin{aligned} e^{Ay} e^{Aw} = e^{A(y+w)} &= \frac{1}{(1-y-w)^{\alpha}} \\ &= \frac{1}{(1-y)^{\alpha}} \frac{1}{\left(1 - \dfrac{w}{1-y}\right)^{\alpha}} \\ &= \frac{1}{(1-y)^{\alpha}} \exp\left(\frac{A}{1-y} w\right). \end{aligned} \tag{3.4}$$

by (3.2). Equating coefficients of $w^k/k!$ shows that the lemma is true for $f(z) = z^k$. The general case then follows by linearity.

Alternatively, we could have introduced an umbra $F$ with $e^{Fz} = f(z)$ and replaced $w$ with $F$ in (3.4). $\square$

As a first application of Lemma 3.1, we prove the following little-known result.

**Theorem 3.2.**

$$\sum_{m=0}^{\infty} C_{2m}(u,\alpha) \frac{x^m}{m!} = e^{u^2 x} \sum_{k=0}^{\infty} \frac{(\alpha)_{2k}}{(1-2ux)^{2k+\alpha}} \frac{x^k}{k!}.$$

*Proof.* We have

$$\begin{aligned}
\sum_{m=0}^{\infty} C_{2m}(u,\alpha)\frac{x^m}{m!} &= \sum_{m=0}^{\infty}(A+u)^{2m}\frac{x^m}{m!} = e^{(A+u)^2x} \\
&= e^{(A^2+2Au+u^2)x} = e^{u^2x}e^{2Aux}e^{A^2x} \\
&= \frac{e^{u^2x}}{(1-2ux)^\alpha}\exp\left[\left(\frac{A}{1-2ux}\right)^2 x\right] \quad \text{by Lemma 3.1} \\
&= e^{u^2x}\sum_{k=0}^{\infty}\frac{(\alpha)_{2k}}{(1-2ux)^{2k+\alpha}}\frac{x^k}{k!}.
\end{aligned}$$

□

By a similar computation we can prove a generalization given by the next theorem. We leave the details to the reader.

**Theorem 3.3.**

$$\sum_{m,n=0}^{\infty} C_{2m+n}(u,\alpha)\frac{x^m}{m!}\frac{y^n}{n!} = e^{u^2x+uy}\sum_{k=0}^{\infty}\frac{(\alpha)_{2k}}{(1-2ux-y)^{2k+\alpha}}\frac{x^k}{k!}. \quad \square$$

Next we prove the bilinear generating function for Charlier polynomials. An equivalent formula can be found in Askey [3, p. 16, equation (2.47)] with a minor error; $a$ and $b$ must be switched on one side of the formula as given there for it to be correct. A combinatorial proof of our Theorem 3.4 has been given by Jayawant [28], who also proved a multilinear generalization.

**Theorem 3.4.**

$$\sum_{n=0}^{\infty} C_n(u,\alpha)C_n(v,\beta)\frac{x^n}{n!} = e^{uvx}\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(1-vx)^{k+\alpha}}\frac{(\beta)_k}{(1-ux)^{k+\beta}}\frac{x^k}{k!}.$$

*Proof.* Let $A$ and $B$ be independent umbrae with $A^n = (\alpha)_n$ and $B^n = (\beta)_n$. Then there is an analogue of Lemma 3.1 with $B$ replacing $A$ and $\beta$ replacing $\alpha$.

We have

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n(u,\alpha)C_n(v,\beta)\frac{x^n}{n!} &= e^{(A+u)(B+v)x} \\
&= e^{uvx}e^{Avx}e^{(Bu+AB)x} \\
&= e^{uvx}\frac{1}{(1-vx)^\alpha}\exp\left(Bux + \frac{A}{1-vx}Bx\right) \quad \text{by Lemma 3.1} \\
&= \frac{e^{uvx}}{(1-vx)^\alpha}e^{Bux}\exp\left(\frac{A}{1-vx}Bx\right) \\
&= \frac{e^{uvx}}{(1-vx)^\alpha}\cdot\frac{1}{(1-ux)^\beta}\exp\left(\frac{A}{1-vx}\cdot\frac{B}{1-ux}x\right) \quad \text{by Lemma 3.1}
\end{aligned}$$

$$= e^{uvx} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(1-vx)^{k+\alpha}} \frac{(\beta)_k}{(1-ux)^{k+\beta}} \frac{x^k}{k!}.$$

□

The polynomials $C_n(u,\alpha)$ have a simple interpretation in terms of permutation enumeration: the coefficient of $\alpha^i u^j$ in $C_n(u-\alpha,\alpha)$ is the number of permutations of $\{1,2,\dots,n\}$ with $j$ fixed points and $i$ cycles of length at least 2. This follows easily from the exponential generating function

$$e^{ux}\left(\frac{e^{-x}}{1-x}\right)^{\alpha} = \sum_{n=0}^{\infty} C_n(u-\alpha,\alpha)\frac{x^n}{n!}.$$

(See, for example, Stanley [39, chapter 5].) In particular, $C_n(-1,1)$ is the derangement number $D_n$, the number of permutations of $\{1,2,\dots,n\}$ with no fixed points, and Theorems 3.2 and 3.4 give the formulas

$$\sum_{m=0}^{\infty} D_{2m}\frac{x^m}{m!} = e^x \sum_{k=0}^{\infty} \frac{(2k)!}{(1+2x)^{2k+1}}\frac{x^k}{k!}$$

and

$$\sum_{n=0}^{\infty} D_n^2 \frac{x^n}{n!} = e^x \sum_{k=0}^{\infty} \frac{k!}{(1+x)^{2k+2}} x^k.$$

Theorem 3.4 can be generalized to a formula involving 3-line Latin rectangles. See [21] for a combinatorial proof that also uses umbral methods. A more general result was given using the same technique by Zeng [45], and using very different techniques by Andrews, Goulden, and Jackson [2].

## 4. Hermite polynomials

We now prove some similar formulas for Hermite polynomials. Perhaps surprisingly, the proofs are a little harder than those for Charlier polynomials. We first define the umbra $M$ by

$$e^{Mx} = e^{-x^2}, \tag{4.1}$$

so that

$$M^n = \begin{cases} (-1)^k \dfrac{(2k)!}{k!}, & \text{if } n = 2k \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

(The reason for the minus sign in this definition is so that we can obtain formulas for the Hermite polynomials in their usual normalization.) There are two basic simplification formulas for $M$:

**Lemma 4.1.**

(i) $e^{M^2x} = \dfrac{1}{\sqrt{1+4x}}$.

(ii) *For any admissible formal power series $f$, we have*

$$e^{My} f(M) = e^{-y^2} f(M - 2y).$$

*Proof.* For (i), we have

$$e^{M^2 x} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{k!} \frac{x^k}{k!} = \frac{1}{\sqrt{1+4x}}.$$

For (ii), as in the proof of Lemma 3.1, it is sufficient to prove that the formula holds for $f(z) = e^{zw}$. In this case we have

$$e^{My} e^{Mw} = e^{M(y+w)} = e^{-y^2-2yw-w^2} = e^{-y^2} e^{-2yw} e^{Mw} = e^{-y^2} e^{(M-2y)w}.$$

□

**Lemma 4.2.**

$$e^{Mx+M^2 y} = \frac{e^{-x^2/(1+4y)}}{\sqrt{1+4y}}.$$

Although Lemma 4.2 can be proved directly by showing that both sides are equal to

$$\sum_{i,j} (-1)^{j+k} \frac{(2j+2k)!}{(j+k)!} \frac{x^{2j}}{(2j)!} \frac{y^k}{k!},$$

we give instead two proofs that use Lemma 4.1

*First proof.* If we try to apply Lemma 4.1 directly, we find that the linear term in $M$ does not disappear, so we need to use a slightly less direct approach. We write $e^{Mx+M^2y}$ as $e^{M(x+z)} e^{M^2 y - Mz}$, where $z$ will be chosen later. Now applying Lemma 4.1 gives

$$\begin{aligned} e^{M(x+z)} e^{M^2 y - Mz} &= e^{-(x+z)^2} e^{(M-2x-2z)^2 y - (M-2x-2z)z} \\ &= e^{-(x+z)^2} e^{M^2 y - 4M(x+z)y + 4(x+z)^2 y - Mz + (2x+2z)z}. \end{aligned}$$

We now choose $z$ so as to eliminate the linear term in $M$ on the right; i.e., we want $-4(x+z)y - z = 0$. So we take $z = -4xy/(1+4y)$, and on simplifying we obtain $e^{Mx+M^2y} = e^{-x^2/(1+4y)+M^2 y}$. Then applying Lemma 4.1 (i) gives the desired result. □

*Second proof.* Let us fix $y$ and set $g(x) = e^{Mx+M^2y}$. Applying Lemma 4.1 directly gives

$$\begin{aligned} g(x) = e^{Mx+M^2y} &= e^{-x^2} e^{(M-2x)^2 y} \\ &= e^{-x^2+4x^2 y} e^{-4Mxy+M^2 y} = e^{-x^2(1-4y)} g(-4xy). \end{aligned}$$

Iterating and taking a limit yields

$$\begin{aligned} g(x) &= e^{-x^2(1-4y) - 4^2 x^2 y^2 (1-4y) - \cdots} = e^{-x^2(1-4y+4^2y^2-4^3y^3+\cdots)} g(0) \\ &= e^{-x^2/(1+4y)} g(0) = e^{-x^2/(1+4y)} / \sqrt{1+4y} \end{aligned}$$

by Lemma 4.1 (i). □

Now we define the Hermite polynomials $H_n(u)$ by the generating function

$$\sum_{n=0}^{\infty} H_n(u)\frac{x^n}{n!} = e^{2ux-x^2} = e^{(2u+M)x} \tag{4.2}$$

so that $H_n(u) = (2u+M)^n$.

First we prove a well-known analogue of Theorem 3.2, a special case of a result of Doetsch [10, equation (10)].

**Theorem 4.3.**

$$\sum_{n=0}^{\infty} H_{2n}(u)\frac{x^n}{n!} = \frac{1}{\sqrt{1+4x}}\exp\left(\frac{4u^2x}{1+4x}\right).$$

*Proof.* We have

$$\begin{aligned}\sum_{n=0}^{\infty} H_{2n}(u)\frac{x^n}{n!} &= e^{(2u+M)^2x} = e^{4u^2x}e^{4Mux+M^2x}\\ &= \frac{1}{\sqrt{1+4x}}e^{4u^2x}\exp\left(\frac{-16u^2x^2}{1+4x}\right) \quad \text{by Lemma 4.2}\\ &= \frac{1}{\sqrt{1+4x}}\exp\left(\frac{4u^2x}{1+4x}\right).\end{aligned}$$

□

By the same reasoning we can prove the following generalization of Theorem 4.3.

**Theorem 4.4.**

$$\sum_{m,n=0}^{\infty} H_{2m+n}(u)\frac{x^m}{m!}\frac{y^n}{n!} = \frac{1}{\sqrt{1+4x}}\exp\left(\frac{4u^2x+2uy-y^2}{1+4x}\right). \quad \square$$

Equating coefficients of $y^n/n!$ in both sides of Theorem 4.4, and using (4.2) yields

$$\sum_{m=0}^{\infty} H_{2m+n}(u)\frac{x^m}{m!} = (1+4x)^{-(n+1)/2}H_n\left(\frac{u}{\sqrt{1+4x}}\right)\exp\left(\frac{4u^2x}{1+4x}\right),$$

which is the general form of Doetsch's result [10].

We state without proof a "triple" version of Theorem 4.3 that can be proved by the same technique. See Jayawant [28], where umbral and combinatorial proofs are given.

**Theorem 4.5.**

$$\sum_{n=0}^{\infty} H_{3n}(u)\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}}\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\frac{x^{2n}}{(2n)!},$$

*where* $v = \left(\sqrt{1+48ux}-1\right)/(24x)$.

□

Next we prove Mehler's formula, which gives a bilinear generating function for the Hermite polynomials. An elegant combinatorial proof of this formula has been given by Foata [13], and generalized to the multilinear case by Foata and Garsia [14, 15].

**Theorem 4.6.**

$$\sum_{n=0}^{\infty} H_n(u)H_n(v)\frac{x^n}{n!} = \frac{1}{\sqrt{1-4x^2}} \exp\left(4\frac{uvx-(u^2+v^2)x^2}{1-4x^2}\right).$$

*Proof.* We use two independent umbrae, $M$ and $N$, with $M$ as before and $N^n = M^n$ for all $n$. (In the terminology of Rota and Taylor [37, 36], $M$ and $N$ are "exchangeable" umbrae.) Then

$$\begin{aligned}
\sum_{n=0}^{\infty} H_n(u)H_n(v)\frac{x^n}{n!} &= e^{(2u+M)(2v+N)x}\\
&= e^{2u(2v+N)x}e^{M(2v+N)x}\\
&= e^{2u(2v+N)x}e^{-(2v+N)^2x^2} \quad \text{by (4.1)}\\
&= e^{4vx(u-vx)}e^{2Nx(u-2vx)-N^2x^2}\\
&= \frac{e^{4vx(u-vx)}}{\sqrt{1-4x^2}} \exp\left(-\frac{4x^2(u-2vx)^2}{1-4x^2}\right) \quad \text{by Lemma 4.2}\\
&= \frac{1}{\sqrt{1-4x^2}} \exp\left(4\frac{uvx-(u^2+v^2)x^2}{1-4x^2}\right).
\end{aligned}$$

□

## 5. Carlitz and Zeilberger's Hermite polynomials

Next we consider analogues of the Hermite polynomials studied by Carlitz [8] and Zeilberger [44]. Carlitz considered the "Hermite polynomials of two variables"

$$H_{m,n}(u,v) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!\, u^{m-k}v^{n-k},$$

with generating function

$$\sum_{m,n} H_{m,n}(u,v)\frac{x^m}{m!}\frac{y^n}{n!} = e^{ux+vy+xy}$$

and proved the bilinear generating function

$$\begin{aligned}
\sum_{m,n=0}^{\infty} & H_{m,n}(u_1,v_1)H_{m,n}(u_2,v_2)\frac{x^m}{m!}\frac{y^n}{n!}\\
&= (1-xy)^{-1}\exp\left(\frac{u_1u_2x+v_1v_2y+(u_1v_1+u_2v_2)xy}{1-xy}\right). \quad (5.1)
\end{aligned}$$

Independently, Zeilberger considered the "straight Hermite polynomials"

$$H_{m,n}(w) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!\, w^k,$$

with generating function

$$\sum_{m,n} H_{m,n}(w)\frac{x^m}{m!}\frac{y^n}{n!} = e^{x+y+wxy},$$

and gave a combinatorial proof, similar to Foata's proof of Mehler's formula [13], of the bilinear generating function

$$\sum_{m,n=0}^{\infty} H_{m,n}(u)H_{m,n}(v)\frac{x^m}{m!}\frac{y^n}{n!} = (1-uvxy)^{-1}\exp\left(\frac{x+y+(u+v)xy}{1-uvxy}\right). \tag{5.2}$$

It is easy to see that Carlitz's and Zeilberger's polynomials are related by $H_{m,n}(u,v) = u^m v^n H_{m,n}(1/uv)$, and that (5.1) and (5.2) are equivalent. We shall prove (5.2), since it involves fewer variables. Our proof uses umbrae that are not independent.

We define the umbrae $A$ and $B$ by

$$A^m B^n = \delta_{m,n} m!,$$

where $\delta_{m,n}$ is 1 if $m = n$ and 0 otherwise. Equivalently, $A$ and $B$ may be defined by

$$e^{Ax+By} = e^{xy}. \tag{5.3}$$

Then Zeilberger's straight Hermite polynomials are given by $H_{m,n}(u) = (1+A)^m(1+Bu)^n$. Two of the basic properties of these umbrae are given in the following lemma.

**Lemma 5.1.**

(i) *If $f(x,y)$ is an admissible power series then $e^{Ar+Bs}f(A,B) = e^{rs}f(A+r,B+s)$.*

(ii) $e^{Ax+By+ABz} = \dfrac{1}{1-z}e^{xy/(1-z)}$.

*Proof.* As in the proof of Lemma 3.1, it is sufficient to prove (i) for the case $f(x,y) = e^{xu+yv}$, and for this case we have

$$e^{Ar+Bs}f(A,B) = e^{Ar+Bs}e^{Au+Bv} = e^{A(r+u)}e^{B(s+v)} = e^{(r+u)(s+v)},$$

by (5.3), and

$$\begin{aligned} e^{rs}f(A+s,B+r) &= e^{rs}e^{(A+s)u+(B+r)v} = e^{rs+rv+su}e^{Au+Bv} \\ &= e^{rs+rv+su}e^{uv} = e^{(r+u)(s+v)}. \end{aligned}$$

We can prove (ii) by using (i) to reduce it to the case $x = y = 0$, but instead we give a direct proof. We have

$$\begin{aligned} e^{Ax+By+ABz} &= \sum_{i,j,k} A^{i+k}B^{j+k}\frac{x^i}{i!}\frac{y^j}{j!}\frac{z^k}{k!} \\ &= \sum_{j,k}(j+k)!\,\frac{(xy)^j}{j!^2}\frac{z^k}{k!} = \sum_{j,k}\binom{j+k}{j}\frac{(xy)^j}{j!}z^k \\ &= \sum_j \frac{1}{(1-z)^{j+1}}\frac{(xy)^j}{j!} = \frac{1}{1-z}e^{xy/(1-z)}. \end{aligned}$$

∎

Now we prove Zeilberger's bilinear generating function (5.2). We introduce two independent pairs of umbrae $A_1, B_1$ and $A_2, B_2$ such that each pair behaves like $A, B$; in other words,

$$A_1^k B_1^l A_2^m B_2^n = \delta_{k,l}\delta_{m,n} k!\, m!.$$

Then

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} H_{m,n}(u)H_{m,n}(v)\frac{x^m}{m!}\frac{y^n}{n!}\\
&\qquad = \sum_{m,n}(1+A_1)^m(1+B_1u)^n(1+A_2)^m(1+B_2v)^n\frac{x^m}{m!}\frac{y^n}{n!}\\
&\qquad = e^{(1+A_1)(1+A_2)x+(1+B_1u)(1+B_2v)y}\\
&\qquad = e^{(1+A_2)x+(1+B_2v)y}e^{A_1(1+A_2)x+B_1(1+B_2v)uy}.
\end{aligned}$$

Applying (5.3) with $A_1$ and $B_1$ for $A$ and $B$ yields

$$e^{(1+A_2)x+(1+B_2v)y+(1+A_2)(1+B_2v)uxy} = e^{x+y+uxy}e^{A_2x(1+uy)+B_2vy(1+ux)+A_2B_2uvxy}.$$

Then applying Lemma 5.1 (ii) yields (5.2).

By similar reasoning, we can prove a generating function identity equivalent to the Pfaff-Saalschütz theorem for hypergeometric series [22]. In terms of Carlitz's Hermite polynomials of two variables, this is the evaluation of

$$\sum_{m,n} H_{m,n+j}(0,1)H_{m+i,n}(0,1)\frac{x^m}{m!}\frac{y^n}{n!}.$$

**Theorem 5.2.** *Let $i$ and $j$ be nonnegative integers. Then*

$$\sum_{m,n=0}^{\infty}\binom{m+i}{n}\binom{n+j}{m}x^my^n = \frac{(1+x)^j(1+y)^i}{(1-xy)^{i+j+1}}. \tag{5.4}$$

*Proof.* With $A_1$, $B_1$, $A_2$, and $B_2$ as before, we have

$$A_1^m(1+B_1)^{n+j}A_2^n(1+B_2)^{m+i} = m!\,n!\binom{m+i}{n}\binom{n+j}{m},$$

so the left side of (5.4) is equal to

$$\begin{aligned}
&\sum_{m,n} A_1^m(1+B_1)^{n+j}A_2^n(1+B_2)^{m+i}\frac{x^m}{m!}\frac{y^n}{n!}\\
&\qquad\qquad = e^{A_1(1+B_2)x+A_2(1+B_1)y}(1+B_2)^i(1+B_1)^j. 
\end{aligned}\tag{5.5}$$

Multiplying the right side of (5.5) by $u^iv^j/i!\,j!$, and summing on $i$ and $j$, we obtain

$$e^{A_1(1+B_2x)+A_2(1+B_1)y+(1+B_2)u+(1+B_1)v} = e^{u+v}e^{A_1(1+B_2)x+B_1(v+A_2y)+A_2y+B_2u}.$$

Applying (5.3), with $A_1$ and $B_1$ for $A$ and $B$, gives

$$e^{u+v}e^{(1+B_2)(v+A_2y)x+A_2y+B_2u} = e^{u+v+xv}e^{A_2(1+x)y+B_2(u+xv)+A_2B_2xy}.$$

Applying Lemma 5.1 (ii), we obtain

$$\frac{e^{u+v+xv}}{1-xy}\exp\left(\frac{(1+x)(u+xv)y}{1-xy}\right)=\frac{1}{1-xy}\exp\left(\frac{(1+x)v+(1+y)u}{1-xy}\right),$$

and extracting the coefficient of $u^i v^j/i!\,j!$ gives the desired result. $\square$

We can also prove analogues of Doetsch's theorem (Theorem 4.3) for the straight Hermite polynomials. We need the following lemma, which enables us to evaluate the exponential of any quadratic polynomial in $A$ and $B$.

**Lemma 5.3.**

$$e^{Av+Bw+A^2x+ABy+B^2z}=\frac{1}{\sqrt{(1-y)^2-4xz}}\exp\left(\frac{vw(1-y)+v^2z+w^2x}{(1-y)^2-4xz}\right).$$

*Proof.* Since the proof is similar to earlier proofs, we omit some of the details. The case $v=w=0$ is easy to prove directly. For the general case, we write $e^{Av+Bw+A^2x+ABy+B^2z}$ as $e^{Ar+Bs}\cdot e^{-Ar-Bs+Av+Bw+A^2x+ABy+B^2z}$ and choose $r$ and $s$ so that when Lemma 5.1 (i) is applied, the linear terms in $A$ and $B$ vanish. We find that the right values for $r$ and $s$ are

$$r=\frac{v(1-y)+2wx}{(1-y)^2-4xz}\quad\text{and}\quad s=\frac{w(1-y)+2vz}{(1-y)^2-4xz},$$

and the result of the substitution is

$$\exp\left(\frac{vw(1-y)+v^2z+w^2x}{(1-y)^2-4xz}\right)e^{A^2x+ABy+B^2z},$$

which may be evaluated by the case $v=w=0$. $\square$

**Theorem 5.4.**

$$\sum_{m,n=0}^{\infty}H_{2m,n}(u)\frac{x^m}{m!}\frac{y^n}{n!}=e^{x+y+2uxy+u^2xy^2}$$

$$\sum_{m,n=0}^{\infty}H_{2m,2n}(u)\frac{x^m}{m!}\frac{y^n}{n!}=\frac{1}{\sqrt{1-4u^2xy}}\exp\left(\frac{x+y+4uxy}{1-4u^2xy}\right)$$

$$\sum_{m=0}^{\infty}H_{m,m}(u)\frac{x^m}{m!}=\frac{1}{1-ux}\exp\left(\frac{x}{1-ux}\right)$$

*Proof.* For the first formula, we have

$$\sum_{m,n=0}^{\infty}H_{2m,n}(u)\frac{x^m}{m!}\frac{y^n}{n!}=\sum_{m,n=0}^{\infty}(1+A)^{2m}(1+uB)^n\frac{x^m}{m!}\frac{y^n}{n!}=e^{(1+A)^2x+(1+uB)y}.$$

We simplify this with Lemma 5.3. The proofs of the other two formulas are similar. (The third formula is equivalent to a well-known generating function for Laguerre polynomials.) $\square$

By the same reasoning, we can prove a more general formula that includes all three formulas of Theorem 5.4 as special cases.

**Theorem 5.5.**

$$\sum_{i,j,k,l,m=0}^{\infty} H_{i+2k+m,j+2l+m}(u)\frac{v^i}{i!}\frac{w^j}{j!}\frac{x^k}{k!}\frac{y^l}{l!}\frac{z^m}{m!} = \frac{1}{\sqrt{(1-uz)^2-4u^2xy}}$$
$$\times \exp\left(\frac{(1+uw)^2x+(1+uv)^2y+4uxy+(1-uz)(v+w+z+uvw)}{(1-uz)^2-4u^2xy}\right). \quad \square \tag{5.6}$$

## 6. Rogers-Szegő polynomials

Next we give a proof of a bilinear generating function for the Rogers-Szegő polynomials, which are closely related to $q$-Hermite polynomials. Our proof differs from the other proofs in this paper in that it uses a linear functional on a noncommutative polynomial algebra. A traditional proof of this result can be found in Andrews [1; p. 50, Example 9] which is also a good reference for basic facts about $q$-series.

In this section we let $(a)_m$ denote the $q$-factorial

$$(a)_m = (1-a)(1-aq)\cdots(1-aq^{m-1}),$$

with $(a)_\infty = \lim_{m\to\infty}(a)_m$ as a power series in $q$. In particular,

$$(q)_m = (1-q)(1-q^2)\cdots(1-q^m).$$

The $q$-binomial coefficient $\left[{n \atop k}\right]$ is defined to be $(q)_n/(q)_k(q)_{n-k}$. The Rogers-Szegő polynomials $R_n(u)$ are defined by

$$R_n(u) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} u^k.$$

We will use a $q$-analogue of the exponential function,

$$e(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q)_n}.$$

We will also need the $q$-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} x^n = \frac{(ax)_\infty}{(x)_\infty};$$

the special case $a=0$ gives

$$e(x) = \frac{1}{(x)_\infty},$$

from which it follows that $e(q^j x) = (x)_j e(x)$.

If $A$ and $B$ are noncommuting variables satisfying the commutation relation $BA = qAB$, then it is well known that

$$(A+B)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} A^k B^{n-k}, \tag{6.1}$$

and it follows easily from (6.1) that $e\big((A+B)x\big) = e(Ax)e(Bx)$, where $x$ commutes with $A$ and $B$. We shall also need the easily-proved fact that $B^jA^i = q^{ij}A^iB^j$.

Now let $A, B, C$, and $D$ be noncommuting variables such that $BA = qAB$, $DC = qCD$, and all other pairs of variables commute. We shall work in the ring of formal power series in $A$, $B$, $C$, $D$, with our ring of scalars (which commute with everything) containing variables $u$, $v$, $x$ and $q$. We define our evaluation functional by $\operatorname{eval}(A^iB^jC^kD^l) = u^iv^k$.

Since we need to do some of our computations in the ring of formal power series in $A$, $B$, $C$, and $D$, we write out the applications of eval explicitly in this proof.

**Theorem 6.1.**

$$\sum_{n=0}^{\infty} R_n(u)R_n(v)\frac{x^n}{(q)_n} = \frac{(uvx^2)_\infty}{(uvx)_\infty(ux)_\infty(vx)_\infty(x)_\infty}.$$

*Proof.* By (6.1),

$$\operatorname{eval}\left(\sum_{n=0}^{\infty}(A+B)^n(C+D)^n\frac{x^n}{(q)_n}\right) = \sum_{n=0}^{\infty} R_n(u)R_n(v)\frac{x^n}{(q)_n}. \tag{6.2}$$

Also, we have

$$\begin{aligned}\sum_{n=0}^{\infty}(A+B)^n(C+D)^n\frac{x^n}{(q)_n} &= e\big((A+B)(C+D)x\big)\\ &= e\big(A(C+D)x\big)e\big(B(C+D)x\big)\\ &= e(ACx)e(ADx)e(BCx)e(BDx). \end{aligned}\tag{6.3}$$

The only variables "out of order" in this product are the $D$'s and $C$'s in $e(ADx)e(BCx)$, so

$$\begin{aligned}&\operatorname{eval}\big(e(ACx)e(ADx)e(BCx)e(BDx)\big)\\ &\qquad= \operatorname{eval}\big(e(ACx)\big)\operatorname{eval}\big(e(ADx)e(BCx)\big)\operatorname{eval}\big(e(BDx)\big)\\ &\qquad= e(uvx)\operatorname{eval}\big(e(ADx)e(BCx)\big)e(x)\\ &\qquad= \frac{\operatorname{eval}\big(e(ADx)e(BCx)\big)}{(x)_\infty(uvx)_\infty}. \end{aligned}\tag{6.4}$$

We have

$$e(ADx)e(BCx) = \sum_{i,j=0}^{\infty}\frac{(ADx)^i}{(q)_i}\frac{(BCx)^j}{(q)_j} = \sum_{i,j=0}^{\infty}\frac{A^iB^jC^jD^iq^{ij}x^{i+j}}{(q)_i(q)_j},$$

so

$$\begin{aligned}\operatorname{eval}\big(e(ADx)e(BCx)\big) &= \sum_{i,j=0}^{\infty}\frac{u^iv^jq^{ij}x^{i+j}}{(q)_i(q)_j} = \sum_{i=0}^{\infty}\frac{(ux)^i}{(q)_i}\sum_{j=0}^{\infty}\frac{(vxq^i)^j}{(q)_j}\\ &= \sum_{i=0}^{\infty}\frac{(ux)^i}{(q)_i(vxq^i)_\infty} = \frac{1}{(vx)_\infty}\sum_{i=0}^{\infty}\frac{(vx)_i}{(q)_i}(ux)^i\\ &= \frac{1}{(vx)_\infty}\frac{(uvx^2)_\infty}{(ux)_\infty}. \end{aligned}\tag{6.5}$$

The theorem then follows from (6.2), (6.3), (6.4), and (6.5). $\square$

It is worth pointing out that although our proof uses noncommuting variables, it does not yield a noncommutative generalization of the result, since the last application of eval is necessary for the final simplification.

## 7. Bernoulli numbers

The Bernoulli numbers $B_n$ are defined by the exponential generating function

$$B(x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}. \tag{7.1}$$

Since (7.1) implies that $e^x B(x) = x + B(x)$, the Bernoulli umbra $B$ defined by $B^n = B_n$ satisfies

$$(B+1)^n = B^n + \delta_{n-1}, \tag{7.2}$$

where $\delta_m$ is 1 if $m = 0$ and is 0 otherwise. From (7.2) it follows by linearity that for any admissible formal power series $f$,

$$f(B+1) = f(B) + f'(0). \tag{7.3}$$

Formula (7.3) may be iterated to yield

$$f(B+k) = f(B) + f'(0) + f'(1) + \cdots + f'(k-1) \tag{7.4}$$

for any nonnegative integer $k$.

There are three other important basic identities for the Bernoulli umbra. Although the most straightforward proofs use exponential generating functions, the umbral proofs are interesting and are therefore included here. Very different umbral proofs of these identities have been given by Rota and Taylor [37, Theorem 4.2 and Proposition 8.3].

**Theorem 7.1.**

(i) $(B+1)^n = (-B)^n$.

(ii) $(-B)^n = B^n$ *for* $n \neq 1$, *with* $B_1 = -\frac{1}{2}$. *Thus* $B_n = 0$ *when* $n$ *is odd and greater than 1.*

(iii) *For any positive integer* $k$,

$$kB^n = (kB)^n + (kB+1)^n + \cdots + (kB+k-1)^n.$$

*Proof.* We prove "linearized" versions of these formulas: for any polynomial $f$, we have

$$f(B+1) = f(-B) \tag{7.5}$$

$$f(-B) = f(B) + f'(0) \tag{7.6}$$

$$kf(B) = f(kB) + f(kB+1) + \cdots + f(kB+k-1) \tag{7.7}$$

First note that (7.6) follows immediately from (7.5) and (7.3). We prove (7.5) and (7.7) by choosing polynomials $f(x)$, one of each possible degree, for which the formula to be proved is an easy consequence of (7.3).

For (7.5), we take $f(x) = x^n - (x-1)^n$, where $n \geq 1$. Then

$$f(B+1) = (B+1)^n - B^n = \delta_{n-1} \quad \text{by (7.2)},$$

and since $f(-x) = (-1)^{n-1} f(x+1)$, we have

$$f(-B) = (-1)^{n-1} f(B+1) = (-1)^{n-1}\delta_{n-1} = \delta_{n-1} = f(B+1).$$

For (7.7), we take $f(x) = (x+1)^n - x^n$, where $n \geq 1$. Then $f(B) = \delta_{n-1}$ and

$$\begin{aligned}\sum_{i=0}^{k-1} f(kB+i) &= \sum_{i=0}^{k-1} (kB+i+1)^n - \sum_{i=0}^{k-1} (kB+1)^n \\ &= (kB+k)^n - (kB)^n = k^n\big((B+1)^n - B^n\big) \\ &= k^n \delta_{n-1} = k\delta_{n-1} = kf(B).\end{aligned}$$

□

For later use, we note two consequences of Theorem 7.1. First, combining (7.4) and (7.6) gives

$$f(B+k) - f(-B) = \sum_{i=1}^{k-1} f'(i). \tag{7.8}$$

Second, suppose that $f(u)$ is a polynomial satisfying $f(u+1) = f(-u)$. Then we have

$$\begin{aligned}f(B) &= \tfrac{1}{2}\big(f(2B) + f(2B+1)\big) \quad \text{by (7.7)} \\ &= \tfrac{1}{2}\big(f(2B) + f(-2B)\big) \\ &= f(2B) + f'(0) \quad \text{by (7.6)}.\end{aligned} \tag{7.9}$$

Next, we discuss an identity of Kaneko [29], who set $\tilde{B}_n = (n+1)B_n$ and gave the identity

$$\sum_{i=0}^{n+1} \binom{n+1}{i} \tilde{B}_{n+i} = 0, \tag{7.10}$$

noting that it (together with the fact that $B_{2j+1} = 0$ for $j > 0$) allows the computation of $B_{2n}$ from only half of the preceding Bernoulli numbers. Kaneko's proof is complicated, though his paper also contains a short proof by D. Zagier. We shall show that Kaneko's identity is a consequence of the following nearly trivial result.

**Lemma 7.2.** *For any nonnegative integers $m$ and $n$,*

$$\sum_{i=0}^{m} \binom{m}{i} B_{n+i} = (-1)^{m+n} \sum_{j=0}^{n} \binom{n}{j} B_{m+j}.$$

*Proof.* Take $f(x) = x^m(x-1)^n$ in (7.5). □

The key to Kaneko's identity is the observation that

$$\binom{n+1}{i} \tilde{B}_{n+i} = (n+1)\left[\binom{n+1}{i} + \binom{n}{i-1}\right] B_{n+i}, \tag{7.11}$$

which reveals that (7.10) is simply the case $m = n+1$ of Lemma 7.2.

We can generalize Kaneko's identity in the following way:

**Theorem 7.3.**

$$\frac{1}{n+1}\sum_{i=0}^{n+1} 2^{n+1-i}\binom{n+1}{i}\tilde{B}_{n+i} = (-1)^n,$$

$$\frac{1}{n+1}\sum_{i=0}^{n+1} 3^{n+1-i}\binom{n+1}{i}\tilde{B}_{n+i} = (-2)^{n-1}(n-4),$$

$$\frac{1}{n+1}\sum_{i=0}^{n+1} 4^{n+1-i}\binom{n+1}{i}\tilde{B}_{n+i} = (-1)^n\bigl(4^n + (2-\tfrac{4}{3}n)3^n\bigr),$$

*and in general,*

$$\frac{1}{n+1}\sum_{i=0}^{n+1} k^{n+1-i}\binom{n+1}{i}\tilde{B}_{n+i} = \sum_{i=1}^{k-1}\bigl((2n+1)i-(n+1)k\bigr)i^n(i-k)^{n-1}. \tag{7.12}$$

*Proof.* Using (7.11), we see that the left side of (7.12) is

$$(B+k)^{n+1}B^n + B^{n+1}(B+k)^n.$$

Setting $f(x) = x^m(x-k)^n$ in (7.8), we have

$$(B+k)^mB^n - (-1)^{m+n}B^m(B+k)^n = \sum_{i=0}^{k-1}\bigl((m+n)i-km\bigr)i^{m-1}(i-k)^{n-1}.$$

Setting $m = n+1$ gives (7.12). □

There are several interesting identities for Bernoulli numbers that actually hold for any two sequences $(c_n)$ and $(d_n)$ related umbrally by $d^n = (c+1)^n$; i.e.,

$$d_n = \sum_{i=0}^{n}\binom{n}{i}c_i. \tag{7.13}$$

We note that (7.13) may inverted to give $c^n = (d-1)^n$; i.e.,

$$c_n = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}d_i.$$

By Theorem 7.1 (i), (7.13) holds with $c_n = B_n$, $d_n = (-1)^nB_n$. We shall next describe several pairs of sequences satisfying (7.13), and then give some identities for such sequences, which seem to be new.

Since (7.13) is equivalent to

$$\sum_{n=0}^{\infty} d_n\frac{x^n}{n!} = e^x\sum_{n=0}^{\infty} c_n\frac{x^n}{n!},$$

it is easy to find sequences satisfying (7.13) with simple exponential generating functions, though not all of our examples are of this form.

The derangement numbers $D_n$ satisfy $n! = \sum_{i=0}^{n} \binom{n}{i} D_i$ so (7.13) holds with $c_n = D_n$, $d_n = n!$.

For any fixed nonnegative integer $m$, the Stirling numbers of the second kind $S(m,n)$ satisfy $n^m = \sum_{i=0}^{n} \binom{n}{i} i!\, S(m,i)$, so (7.13) holds with $c_n = n!\, S(m,n)$, $d_n = n^m$.

The Euler numbers $E_n$ are defined by $\sum_{n=0}^{\infty} E_n x^n/n! = \operatorname{sech} x$. Let us define the "signed tangent numbers" $T_n$ by $\tanh x = \sum_{n=0}^{\infty} T_n x^n/n!$. Then since $e^x \operatorname{sech} x = 1 + \tanh x$, we have that (7.13) holds with $c_n = E_n$, $d_n = \delta_n + T_n$.

The Genocchi numbers $g_n$ are defined by $\sum_{n=0}^{\infty} g_n x^n/n! = 2x/(e^x+1)$. Then

$$\frac{2xe^x}{e^x+1} = 2x - \frac{2x}{e^x+1},$$

so (7.13) holds with $c_n = g_n$, $d_n = 2\delta_{n-1} - g_n$ (so that $d_1 = g_1 = 1$).

The Eulerian polynomials $A_n(t)$ satisfy

$$\sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!} = \frac{1-t}{1-te^{(1-t)x}}.$$

Then $A_0 = 1$, and $A_n(t)$ is divisible by $t$ for $n > 1$. Let us set $\tilde{A}_n(t) = t^{-1} A_n(t)$ for $n > 0$, with $\tilde{A}_0(t) = 1$. It is easy to check that

$$e^{(1-t)x} \sum_{n=0}^{\infty} \tilde{A}_n(t) \frac{x^n}{n!} = \sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!},$$

so (7.13) holds with $c_n = A_n(t)/(1-t)^n$, $d_n = \tilde{A}_n(t)/(1-t)^n$.

The Fibonacci numbers $F_n$ are defined by $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n$. It is easily verified that for every fixed integer $m$, (7.13) holds with $c_n = F_{m+n}$, $d_n = F_{m+2n}$, and also with $c_n = F_{m-n}$, $d_n = F_{m+n}$.

By the Chu-Vandermonde theorem, (7.13) holds with

$$c_n = (-1)^n \frac{(\alpha)_n}{(\beta)_n}, \quad d_n = \frac{(\beta-\alpha)_n}{(\beta)_n},$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$.

As Zagier observed [29], it is easy to characterize the pairs of sequences satisfying (7.13) with $d_n = (-1)^n c_n$, which, as we shall see, give analogues of Kaneko's identity. The condition, with $c(x) = e^{cx}$ and $d(x) = e^{dx}$, is $e^x c(x) = c(-x)$, which is equivalent to $e^{x/2} c(x) = e^{-x/2} c(-x)$; i.e., $e^{x/2} c(x)$ is even. Thus it is easy to construct such sequences, but not many seem natural. In addition to the Bernoulli numbers, we have an example with the Genocchi numbers $g_n$,

$$c(x) = \frac{2e^{-x/2}}{e^{x/2} + e^{-x/2}} = \frac{2}{e^x+1} = \sum_{n=0}^{\infty} \frac{g_{n+1}}{n+1} \frac{x^n}{n!},$$

and one with the Lucas numbers, $c_n = (-2)^{-n}(L_n + L_{2n})$, where $L_n = F_{n+1} + F_{n-1}$,

We now discuss the identities which are consequences of (7.13). Our first identity generalizes Lemma 7.2.

**Theorem 7.4.** *Suppose that the sequences $c_n$ and $d_n$ satisfy* (7.13). *Then for all nonnegative integers $m$ and $n$,*

$$\sum_{i=0}^{m}\binom{m}{i}c_{n+i}=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}d_{m+j}. \tag{7.14}$$

*Proof.* Let $c$ and $d$ be umbrae with $c^n = c_n$ and $d^n = d_n$. Then (7.13) implies that $(c+1)^n = d^n$, so for any polynomial $f(x)$, we have $f(c+1) = f(d)$. Taking $f(x) = x^m(x-1)^n$ yields the theorem. □

An application of Theorem 7.4 yields an interesting recurrence for Genocchi numbers. Let $c_n$ be the Genocchi number $g_n$, so that, as noted above, $d_n = 2\delta_{n-1} - g_n$. Then taking $m = n$ in Theorem 7.4, we have for $n > 1$,

$$\sum_{i=0}^{n}\binom{n}{i}g_{n+i}=-\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}g_{n+i},$$

so

$$\sum_{i=0}^{n}(1+(-1)^{n-i})\binom{n}{i}g_{n+i}=0.$$

The only nonzero terms in the sum are those with $n - i$ even, so we may set $2j = n - i$ and divide by 2 to get the recurrence

$$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}g_{2n-2j}=0, \quad n>1. \tag{7.15}$$

Equation (7.15) is known as Seidel's recurrence (see, e.g., Viennot [42]). It implies that $g_{2n}$ is an integer, which is not obvious from the generating function (it is easily shown that $g_{2i+1} = 0$ for $i > 0$), and it can also be used to derive a combinatorial interpretation for the Genocchi numbers. The reader can check that the Genocchi analogue of Kaneko's identity alluded to before Theorem 7.4 is also Seidel's recurrence in the form $\sum_{i=0}^{n}\binom{n}{i}g_{n+i} = 0$ (in this form true for all $n$).

We now derive some further identities for sequences satisfying (7.13), of which the first generalizes Theorem 7.4.

**Theorem 7.5.** *Suppose that the sequences $(c_n)$ and $(d_n)$ satisfy $\sum_{i=0}^{n}\binom{n}{i}c_i = d_n$. Then for all $a$ and $b$,*

$$\sum_{i=0}^{n}\binom{a}{i}\binom{b}{n-i}d_i=\sum_{j=0}^{n}\binom{a}{j}\binom{a+b-j}{n-j}c_j \tag{7.16}$$

$$\sum_{i=0}^{n}\binom{a}{i}\binom{2a-2i}{n-i}(-2)^i d_i=\sum_{j=0}^{\lfloor n/2\rfloor}\binom{a}{n-j}\binom{n-j}{j}(-2)^{n-2j}c_{n-2j} \tag{7.17}$$

$$\sum_{i=0}^{n}\binom{a}{i}\binom{2a-2i}{n-i}(-4)^i d_i=(-1)^n\sum_{j=0}^{n}\binom{a}{j}\binom{2a-2j}{n-j}4^j c_j. \tag{7.18}$$

*Proof.* Let $c$ be an umbra with $c^n = c_n$. Then (7.16)–(7.18) follow by substituting $c$ for $u$ in the following polynomial identities, where $v = 1 + u$:

$$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} v^i = \sum_{j=0}^{n} \binom{a}{j}\binom{a+b-j}{n-j} u^j \tag{7.19}$$

$$\sum_{i=0}^{n} \binom{a}{i}\binom{2a-2i}{n-i} (-2v)^i = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{a}{n-j}\binom{n-j}{j} (-2u)^{n-2j} \tag{7.20}$$

$$\sum_{i=0}^{n} \binom{a}{i}\binom{2a-2i}{n-i} (-4v)^i = (-1)^n \sum_{j=0}^{n} \binom{a}{j}\binom{2a-2j}{n-j} (4u)^j. \tag{7.21}$$

We prove (7.19) by extracting the coefficient of $x^n$ in $(1+x)^b(1+vx)^a$ in two ways. It is clear that this coefficient is given by the left side of (7.19). But we also have

$$(1+x)^b(1+vx)^a = (1+x)^b(1+x+ux)^a = (1+x)^{a+b}\left(1+\frac{ux}{1+x}\right)^a,$$

in which the coefficient of $x^n$ is easily seen to be given by the right side of (7.19).

For (7.20), we extract the coefficient of $x^n$ in

$$\big((1+x)^2 - 2xv\big)^a = \big((1+x)^2 - 2x - 2xu\big)^a = (1 + x^2 - 2xu)^a.$$

For the left side we have

$$\begin{aligned}
\big((1+x)^2 - 2xv\big)^a &= (1+x)^{2a}\left(1 - \frac{2xv}{(1+x)^2}\right)^a \\
&= (1+x)^{2a} \sum_i \binom{a}{i} \frac{(-2xv)^i}{(1+x)^{2i}} \\
&= \sum_i \binom{a}{i} x^i (1+x)^{2a-2i} (-2v)^i
\end{aligned}$$

and the coefficient of $x^n$ is the left side of (7.20).

For the right side we have

$$\begin{aligned}
(1 + x^2 - 2xu)^a &= \sum_i \binom{a}{i} (x^2 - 2xu)^i \\
&= \sum_{i,j} \binom{a}{i}\binom{i}{j} x^{2j} (-2xu)^{i-j} \\
&= \sum_{i,j} \binom{a}{i}\binom{i}{j} x^{i+j} (-2u)^{i-j}.
\end{aligned}$$

Setting $i = n - j$ gives the right side of (7.20) as the coefficient of $x^n$.

For (7.21) we start with the identity

$$\big((1+x)^2 - 4xv\big)^a = \big((1+x)^2 - 4x - 4xu\big)^a = \big((1-x)^2 - 4xu\big)^a.$$

The coefficient of $x^n$ may be extracted from both sides as on the left side of (7.20). $\square$

We note that (7.19) is equivalent to a ${}_2F_1$ linear transformation and (7.20) to a ${}_2F_1$ quadratic transformation. Equation (7.21) is actually a special case of (7.19); it can be obtained from (7.19) by replacing $a$ with $2a-n$ and $b$ with $n-a-\frac{1}{2}$, and simplifying.

The special case $a=-1$ of (7.16) is worth noting. It may be written

$$\sum_{i=0}^{n}\binom{b}{n-i}(-1)^i d_i = (-1)^n \sum_{j=0}^{n}\binom{n-b}{n-j}c_j. \tag{7.22}$$

If we replace $n$ by $m+n$ in (7.22) and then set $b=n$, it reduces to (7.14).

Next we prove a remarkable identity of Zagier [43] for Bernoulli numbers. Our proof is essentially an umbral version of Zagier's. The reader may find it instructive to compare the two presentations.

**Theorem 7.6.** *Let*

$$B_n^* = \sum_{r=0}^{n}\binom{n+r}{2r}\frac{B_r}{n+r}$$

*for $n>0$. Then the value of $B_n^*$ for $n$ odd is periodic and is given by*

| $n \pmod{12}$ | 1 | 3 | 5 | 7 | 9 | 11 |
|---|---|---|---|---|---|---|
| $B_n^*$ | $\frac{3}{4}$ | $-\frac{1}{4}$ | $-\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $-\frac{3}{4}$ |

*Proof.* Since $\binom{n+r}{2r}=\binom{n+r-1}{2r-1}\frac{n+r}{2r}$ for $r>0$, we have

$$\begin{aligned}2\sum_{n=1}^{\infty}B_n^*x^n &= 2\sum_{n=1}^{\infty}\left[\frac{1}{n}+\sum_{r=1}^{n}\binom{n+r-1}{2r-1}\frac{B^r}{2r}\right]x^n\\ &= -\log(1-x)^2-\log\left(1-B\frac{x}{(1-x)^2}\right)\\ &= -\log\left((1-x)^2-Bx\right).\end{aligned}$$

Now let $g(u)=-\log(1-ux+x^2)$, so that $2\sum_{n=1}^{\infty}B_n^*x^n = g(B+2)$. Note that

$$g'(u)=\frac{x}{1-ux+x^2}.$$

Taking $k=4$ and $f(u)=g(u-2)$ in (7.8), we have

$$\begin{aligned}g(B+2)-g(-B-2) &= g'(-1)+g'(0)+g'(1)\\ &= \frac{x}{1+x+x^2}+\frac{x}{1+x^2}+\frac{x}{1-x+x^2}\\ &= \frac{3x-x^3-x^5+x^7+x^9-3x^{11}}{1-x^{12}}.\end{aligned}$$

But $g(-B-2)=-\log\left((1+x)^2+Bx\right)=2\sum_{n=1}^{\infty}B_n^*(-x)^n$, so

$$g(B+2)-g(-B-2)=4\sum_{n \text{ odd}}B_n^*x^n,$$

and the result follows. $\square$

## 8. Kummer Congruences

We say that a sequence $(u_n)$ of integers satisfies *Kummer's congruence* for the prime $p$ if for every integer $n$ and every $j \geq n$,

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} u_{i(p-1)+j} \equiv 0 \pmod{p^n}. \tag{8.1}$$

There are many variations and generalizations of this congruence, and we refer the reader to [19], on which most of this section is based, for more information and further references.

If we set $j = n + k$, then (8.1) may be written umbrally as

$$(u^p - u)^n u^k \equiv 0 \pmod{p^n} \tag{8.2}$$

for all $n, k \geq 0$, where $u^m = u_m$.

The result that we prove here shows that if a sequence satisfies Kummer's congruence, then so does the coefficient sequence of the reciprocal of its exponential generating function. Similar results apply to products.

**Theorem 8.1.** *Let $(u_n)$ and $(v_n)$ be sequences of integers satisfying*

$$\left(\sum_{n=0}^{\infty} u_n \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} v_n \frac{x^n}{n!}\right) = 1. \tag{8.3}$$

*Then if $(u_n)$ satisfies Kummer's congruence for the prime $p$, so does $(v_n)$.*

*Proof.* The relation (8.3) may be written umbrally as $(u+v)^n = 0$, for $n > 0$, where $u$ and $v$ are independent umbrae satisfying $u^m = u_m$ and $v^n = v_n$, and this implies that if $f(x)$ is any polynomial with no constant term, then $f(u+v) = 0$. We shall prove by induction that if (8.2) holds for all $n, k \geq 0$ then

$$(v^p - v)^n v^k \equiv 0 \pmod{p^n} \tag{8.4}$$

for all $n, k \geq 0$.

The case $n = 0$ of (8.4) is trivial. Now let $N$ be a positive integer and $K$ a nonnegative integer, and suppose that (8.4) holds whenever $n < N$ and also when $n = N$ but $k < K$. Thus

$$\begin{aligned} 0 &= [(u+v)^p - (u+v)]^N (u+v)^K \\ &= [(u^p - u) + (v^p - v) + pR(u,v)]^N (u+v)^K \end{aligned}$$

for some polynomial $R(u, v)$ with integer coefficients,

$$= (v^p - v)^N v^K + \text{other terms}.$$

Here each other term is an integer times $(u^p - u)^a (v^p - v)^b \big(pR(u,v)\big)^c u^d v^e$, where $a+b+c = N$, $d + e = K$, and either $b < N$ or $b = N$, $c = 0$, and $e < K$. Thus by the inductive hypothesis and (8.2), each of the other terms is divisible by $p^{a+b+c} = p^N$, and therefore $(v^p - v)^N v^K$ is also. $\square$

As an example, we apply Theorem 8.1 to generalized Euler numbers. Recall that the Euler numbers $E_n$ are defined by $\operatorname{sech} x = \sum_{n=0}^{\infty} E_n x^n/n!$ (so $E_n = 0$ when $n$ is odd). We define the generalized Euler numbers $e_n^{(m)}$ by

$$\sum_{n=0}^{\infty} e_n^{(m)} \frac{x^{mn}}{(mn)!} = \left(\sum_{n=0}^{\infty} \frac{x^{mn}}{(mn)!}\right)^{-1},$$

so that $e_n^{(2)} = E_{2n}$.

**Theorem 8.2.** *Let $p$ be a prime and let $m$ be a positive integer such that $d = (p-1)/m$ is an integer. Then for $j \geq n/m$,*

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} e_{id+j}^{(m)} \equiv 0 \pmod{p^n}.$$

*Proof.* Let us take $u_n = 1$ and $v_n = e_{n/m}^{(m)}$ if $m$ divides $n$, with $u_n = v_n = 0$ otherwise. Then the sequences $(u_n)$ and $(v_n)$ satisfy (8.3), and $(u_n)$ satisfy Kummer's congruence for $p$. Therefore $(v_n)$ does also. □

For example, if we take $m = 4$ and $p = 5$ in Theorem 8.2, then $d = 1$ and we have the congruence

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} e_{i+j}^{(4)} \equiv 0 \pmod{5^n} \tag{8.5}$$

for $j \geq n/4$.

By the same kind of reasoning we can prove a variation of Theorem 8.1 [19]:

**Theorem 8.3.** *Let the sequences $(u_n)$ and $(v_n)$ be related by (8.3), and suppose that for some integer $a$,*

$$\sum_{i=0}^{n} a^{n-i} \binom{n}{i} u_{ip+j} \equiv 0 \pmod{p^n}$$

*for all nonnegative integers $j$ and $n$. Then*

$$\sum_{i=0}^{n} (-a)^{n-i} \binom{n}{i} v_{ip+j} \equiv 0 \pmod{p^n}. \quad \square$$

Next we prove a Kummer congruence for Bernoulli numbers. A similar, but weaker, congruence was proved by Carlitz [7] using a different method.

We call a rational number 2-*integral* if its denominator is odd. If $a$ and $b$ are rational numbers, then by $a \equiv b \pmod{2^r}$ we mean that $(a-b)/2^r$ is 2-integral. For example, $\frac{1}{2} \equiv \frac{5}{2} \pmod 2$. We define $\rho_2(a)$ to be the largest integer for which $a/2^{\rho_2(a)}$ is 2-integral; so $\rho_2(\frac{1}{2}) = -1$ and $\rho_2(\frac{4}{3}) = 2$.

In the proof of the next theorem we will use the fact that $2B_n$ is 2-integral for all $n$, and that if $n$ is even and positive then $B_n \equiv \frac{1}{2} \pmod 1$; this follows easily by induction from the case $k = 2$ of Theorem 7.1 (iii).

**Theorem 8.4.** *For nonnegative integers $n$ and $j$,*

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}B_{2i+2j} \equiv 0 \pmod{2^{\tau_{j,n}}}, \tag{8.6}$$

*where $\tau_{0,0}=0$, $\tau_{j,0}=\tau_{0,n}=-1$ for $j>0$ and $n>0$, $\tau_{j,1}=1$ for $j\ge 2$, and*

$$\tau_{j,n} = \min\left(2j-2, 2\left\lfloor \frac{3n-1}{2}\right\rfloor\right)$$

*for $n\ge 2$ and $j\ge 1$. Moreover, the exponent in* (8.6) *is best possible if and only if $j \ne \lfloor (3n+1)/2\rfloor$.*

*Proof.* For simplicity we prove only the most interesting case, in which $n\ge 2$ and $j\ge 1$. Let $B$ be the Bernoulli umbra, $B^n = B_n$, so the sum in (8.6) is $(B^2-1)^n B^{2j}$.

Applying (7.7) with $k=2$ and $f(u)=(u^2-1)^n u^{2j}$, we obtain

$$(B^2-1)^n B^{2j} = 2^{2j-1}(4B^2-1)^n B^{2j} + 2^{2n-1}B^n(B+1)^n(2B+1)^{2j}. \tag{8.7}$$

The first term on the right side of (8.7) is $(-1)^n 2^{2j-1}(B_{2j} - 4nB_{2j+2}+\cdots)$. Since $j>0$, this is congruent to $(-1)^n 2^{2j-2} \pmod{2^{2j}}$ and thus $\rho_2(2^{2j-1}(4B^2-1)^n B^{2j}) = 2j-2$.

Next, let $g(u) = u^n(u+1)^n(2u+1)^{2j}$. To determine $\rho_2\big(g(B)\big)$, we apply (7.3) in the form $g(B) = g(B-1) - g'(-1)$ and we find that (since $n>1$)

$$g(B) = B^n(B+1)^n(2B+1)^{2j} = B^n(B-1)^n(2B-1)^{2j}.$$

We now apply (7.9) to $f(u) = u^n(u-1)^n(2u-1)^{2i}$ and we obtain (since $n>1$)

$$g(B) = f(B) = f(2B) = 2^n B^n(2B-1)^n(4B-1)^{2j} = (-2)^n\big(B_n - 2nB_{n+1} + 2K\big),$$

where $K$ is 2-integral. Thus if $n$ is even,

$$g(B) \equiv 2^n B_n \equiv 2^{n-1} \pmod{2^n},$$

and if $n$ is odd

$$g(B) \equiv 2^{n+1}nB_{n+1} \equiv 2^n \pmod{2^{n+1}}.$$

Thus $\rho_2(g(B))$ is $n-1$ if $n$ is even and $n$ if $n$ is odd; so in either case we have $\rho_2(g(B)) = 2\lfloor (n-1)/2\rfloor + 1$. Thus the power of 2 dividing the second term on the right side of (8.7) is

$$\rho_2(g(B)) = (2n-1) + 2\lfloor (n-1)/2\rfloor + 1 = 2\lfloor (3n-1)/2\rfloor,$$

and the congruence (8.6) follows. It is clear that the exponent in (8.6) is best possible if and only if $2j-2 \ne 2\lfloor (3n-1)/2\rfloor$ and this is equivalent to the stated condition. ∎

We can use Theorem 8.4 to obtain congruences of a different kind for the Bernoulli numbers. As noted earlier, for sequences $(c_n)$ and $(d_n)$ we have $c_n = \sum_{i=0}^n (-1)^{n-i}\binom{n}{i}d_i$ if and only if $d_n = \sum_{i=0}^n \binom{n}{i}c_i$. Let us fix $j>0$ and take $d_n = B_{2n+2j}$, so that $c_n = (B^2-1)^n B^{2j}$. Then we have

$$B_{2n+2j} = \sum_{i=0}^{n}\binom{n}{i}c_i.$$

Moreover, it follows from Theorem 8.4 that if $i \geq (2j-1)/3$ and $i \geq 2$ then $c_i \equiv 0 \pmod{2^{2j-2}}$, so we obtain the congruence

$$B_{2n+2j} \equiv \sum_{i=0}^{M} \binom{n}{i} c_i \pmod{2^{2j-2}}, \tag{8.8}$$

where $M = \max(\lfloor 2(j-1)/3 \rfloor, 1)$. The cases $j = 2, 3, 4$ of (8.8), with simplifications obtained by reducing their coefficients, are

$$\begin{aligned}
B_{2n+4} &\equiv -\frac{1}{30} + \frac{2}{35}n \equiv \frac{1}{2} + 2n \pmod{4}\\
B_{2n+6} &\equiv \frac{1}{42} - \frac{2}{35}n \equiv \frac{13}{2} + 10n \pmod{16}\\
B_{2n+8} &\equiv -\frac{1}{30} + \frac{6}{55}n - \frac{2192}{5005}\binom{n}{2} \equiv \frac{17}{2} + 42n + 48\binom{n}{2} \pmod{64}.
\end{aligned}$$

We note for use in the next section simpler forms of the first two of these congruences:

**Lemma 8.5.** *Let $n$ be an even integer.*

(i) *If $n \geq 4$ then $B_n \equiv \frac{1}{2} + n \pmod{4}$.*

(ii) *If $n \geq 6$ then $B_n \equiv \frac{1}{2} + 5n \pmod{16}$.* □

Of course, more direct proofs of this lemma are possible. Similar congruences for Bernoulli numbers to other moduli have been given by Frame [16]. Many congruences for generalized Euler numbers, obtained in this way from Kummer congruences, can be found in [19].

## 9. Median Genocchi numbers and Kummer congruences for Euler numbers

It follows from Theorem 8.3 that the Euler numbers $E_n$ satisfy the congruence

$$\sum_{i=0}^{n} \binom{n}{i} E_{2i+j} \equiv 0 \pmod{2^n}.$$

However, Frobenius [17] (see also Carlitz [7]) proved a much stronger congruence: the power of 2 dividing

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} E_{2i+j}$$

(for $j$ even) is the same as the power of 2 dividing $2^n n!$. Using the same approach as Frobenius and Carlitz, we prove in this section a Kummer congruence for the numbers $F_n = nE_{n-1}$ in which the modulus is $2^{3n}$ or $2^{3n-1}$, and derive from it Frobenius's congruence.

A special case of our result gives a divisibility property for the *median Genocchi numbers* (also called *Genocchi numbers of the second kind*). These numbers may be defined by

$$H_{2n+1} = \sum_{k=0}^{n} \binom{n}{k} g_{n+k+1},$$

where the $g_i$ are the Genocchi numbers, defined by $\sum_{i=0}^{\infty} g_i x^i/i! = 2x/(e^x+1)$. (In combinatorial investigations, the notation $H_{2n+1}$ is usually used for what in our notation is $|H_{2n+1}| = (-1)^n H_{2n+1}$.) The connection between median Genocchi numbers and the numbers $F_n = nE_{n-1}$ is given by the following result, due to Dumont and Zeng [12].

**Lemma 9.1.**

$$2^{2n} H_{2n+1} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} F_{2i+1} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (2i+1) E_{2i}.$$

*Proof.* Let us define the umbrae $g$ and $F$ by $g^n = g_n$ and $F^n = F_n = nE_{n-1}$, so that $e^{Fx} = x \operatorname{sech} x$. Then

$$e^{gx} = \frac{2x}{e^x+1} = \frac{2xe^{-x/2}}{e^{x/2}+e^{-x/2}} = 2e^{-x/2} \cdot \frac{x}{2} \operatorname{sech} \frac{x}{2} = 2e^{-x/2} e^{\frac{F}{2}x} = 2e^{\frac{1}{2}(F-1)x}.$$

Thus $g^n = 2\left(\frac{F-1}{2}\right)^n$, and it follows by linearity that

$$\begin{aligned} 2^{2n} H_{2n+1} &= 2^{2n}(g+1)^n g^{n+1} = 2^{2n} \cdot 2 \left(\frac{F+1}{2}\right)^n \left(\frac{F-1}{2}\right)^{n+1} \\ &= (F-1)(F^2-1)^n = F(F^2-1)^n, \end{aligned}$$

since $F^m = 0$ for $m$ even. ∎

Barsky [4] proved a conjecture of Dumont that $H_{2n+1}$ is divisible by $2^{n-1}$. More precisely, Barsky proved that for $n \geq 3$, $H_{2n+1}/2^{n-1}$ is congruent to 2 modulo 4 if $n$ is odd and is congruent to 3 modulo 4 if $n$ is even. Kreweras [30] gave a combinatorial proof of Dumont's conjecture, using a combinatorial interpretation of $H_{2n+1}$ due to Dumont [11, Corollaire 2.4]. A $q$-analogue of Barsky's result was given by Han and Zeng [24].

It is interesting to note that (as pointed out in [38]), a combinatorial interpretation of the numbers $H_{2n+1}/2^{n-1}$ was given in 1900 by H. Dellac [9]. Dellac's interpretation may be described as follows: We start with a $2n$ by $n$ array of cells and consider the set $D$ of cells in rows $i$ through $i+n$ of column $i$, for $i$ from 1 to $n$. Then $H_{2n+3}/2^n$ is the number of subsets of $D$ containing two cells in each column and one cell in each row. Dellac did not give any formula for these numbers, but he did compute them for $n$ from 1 to 8. Dellac's interpretation can be derived without too much difficulty from Dumont's combinatorial interpretation, but it is not at all clear how Dellac computed these numbers.

**Theorem 9.2.** *Let $F_n = nE_{n-1}$, so that $\sum_{n=0}^{\infty} F_n x^n/n! = x \operatorname{sech} x$. Let $2^{\mu_{j,n}}$ be the highest power of 2 dividing*

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} F_{2i+j},$$

*where $j$ is odd. Then $\mu_{j,0} = 0$, $\mu_{j,1} = 2$, and for $n > 1$, $\mu_{j,n} = 3n$ if $n$ is odd and $\mu_{j,n} = 3n-1$ if $n$ is even.*

*Proof.* We have

$$\operatorname{sech} x - 2e^{-x} = \frac{2}{e^x+e^{-x}} - \frac{2+2e^{-2x}}{e^x+e^{-x}} = -\frac{2e^{-2x}}{e^x+e^{-x}} = -\frac{2e^{-x}}{e^{2x}+1} = -2\frac{e^x-e^{-x}}{e^{4x}-1}.$$

Therefore

$$x \operatorname{sech} x = 2xe^{-x} - \sinh x \cdot B(4x) \tag{9.1}$$

where $B(x) = x/(e^x - 1)$ is the Bernoulli number generating function.

Now let us define the umbrae $F$, $A$, $B$, and $C$ by

$$\begin{aligned} e^{Fx} &= x \operatorname{sech} x \\ e^{Ax} &= xe^{-x} = \sum_{n=1}^{\infty} (-1)^{n-1} n \frac{x^n}{n!} \\ e^{Bx} &= B(x) = \frac{x}{e^x - 1} \\ e^{Cx} &= \sinh x = \sum_{n \text{ odd}} \frac{x^n}{n!}. \end{aligned}$$

Then from (9.1) we have

$$F^n = 2A^n - (4B + C)^n. \tag{9.2}$$

We want to find the power of 2 dividing $F^j(F^2-1)^n$. It follows from (9.2) that

$$F^j(F^2-1)^n = 2A^j(A^2-1)^n - (4B+C)^j\big((4B+C)^2 - 1\big)^n. \tag{9.3}$$

First note that for any polynomial $p$, $p(A) = p'(-1)$. Therefore,

$$2A^j(A^2-1)^n = \begin{cases} 2(-1)^{j-1} j & \text{if } n = 0 \\ 4(-1)^{j-1} & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases} \tag{9.4}$$

We note also that $C^j(C^2-1)^n = 0$ for all integers $j \geq 0$ and $n \geq 1$. Then

$$\begin{aligned} (4B+C)^j\big((4B+C)^2-1\big)^n &= (4B+C)^j(16B^2 + 8BC + C^2 - 1)^n \\ &= 2^{3n}(4B+C)^j(2B^2+BC)^n. \end{aligned} \tag{9.5}$$

We now need to determine the power of 2 dividing $(4B+C)^j(2B^2+BC)^n$. Since $F^j(F^2-1)^n = 0$ if $j$ is even, we may assume that $j$ is odd. Since $2B_i$ is 2-integral, we have

$$(4B+C)^j(2B^2+BC)^n \equiv C^j(B^nC^n + 2nB^{n+1}C^{n-1}) \pmod{2}.$$

Using the facts that $B_i = 0$ when $i$ is odd and greater than 1, and that

$$C^i = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even,} \end{cases}$$

we find that if $n = 0$ then

$$(4B+C)^j(2B^2+BC)^n \equiv B_0 \equiv 1 \pmod{2},$$

if $n$ is even and positive then

$$(4B+C)^j(2B^2+BC)^n \equiv B_n \equiv \tfrac{1}{2} \pmod{1},$$

and if $n$ is odd then

$$(4B+C)^j(2B^2+BC)^n \equiv 2nB_{n+1} \equiv 1 \pmod 2.$$

The theorem then follows from these congruences, together with (9.3), (9.4), and (9.5). $\square$

By taking more terms in the expansion of $(4B+C)^j(2B^2+BC)^n$, we can get congruences modulo higher powers of 2. For example, if $n$ is even then

$$(4B+C)^j(2B^2+BC)^n \equiv B_n + 4\binom{n}{2}B_{n+2} \pmod 8.$$

Applying Lemma 8.5 (ii), we find that if $n \geq 6$ then

$$B_n + 4\binom{n}{2}B_{n+2} \equiv \tfrac{1}{2} + n^2(n+3) \pmod 8.$$

Since $n$ even implies $n^2 \equiv 2n \pmod 8$ and $4n \equiv 0 \pmod 8$, this simplifies to $\frac{1}{2} - 2n \pmod 8$.

Similarly, if $n$ is odd then

$$(4B+C)^j(2B^2+BC)^n \equiv (2n+4j)B_{n+1} + 4\binom{n}{3} \pmod 8.$$

If $n \geq 3$ then Lemma 8.5 (i) gives $(2n+4j)B_{n+1} \equiv (n+2j)(3+2n) \pmod 8$, and we may easily verify that for $n$ odd, $4\binom{n}{3} \equiv 2n-2 \pmod 8$. Using the fact that $n$ odd implies $n^2 \equiv 1 \pmod 8$ and $4n \equiv 4 \pmod 8$, we obtain

$$(2n+4j)B_{n+1} + 4\binom{n}{3} \equiv 4 + 2j + n \pmod 8.$$

Therefore we may conclude that (for $j$ odd) if $n$ is even and $n \geq 6$ then

$$2^{-(3n-1)} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} F_{2i+j} \equiv 4n - 1 \pmod{16},$$

and if $n$ is odd and $n \geq 3$ then

$$2^{-3n} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} F_{2i+j} \equiv 4 - 2j - n \pmod 8.$$

In particular, we get a refinement of Barsky's theorem: If $n$ is even and $n \geq 6$ then $H_{2n+1}/2^{n-1} \equiv 4n-1 \pmod{16}$, and if $n$ is odd and $n \geq 3$ then $H_{2n+1}/2^n \equiv 2-n \pmod 8$. It is clear that by the same method we could extend these congruences to any power of 2.

Next we derive Frobenius's congruence from Theorem 9.2. (This derivation is similar to part of Frobenius's original proof.) Define the umbra $E$ by $e^{Ex} = \operatorname{sech} x$, so $F^n = nE^{n-1}$. First note that if $j$ is even then $E_j = F_{j+1}/(j+1)$ is odd, so Frobenius's congruence holds for $n=0$. From $F^n = nE^{n-1}$, it follows that for any polynomial $p$, we have $p(F) = p'(E)$. Let us take $p(u) = u^{j+1}(u^2-1)^n$, where $j$ is even and $n \geq 1$. Then

$p'(u) = (j+1)u^j(u^2-1)^n + 2nu^{j+2}(u^2-1)^{n-1}$. By Theorem 9.2, we have $p(F) \equiv 0 \pmod{2^{3n-1}}$, so

$$E^j(E^n-1)^n \equiv -\frac{2n}{j+1}E^{j+2}(E^2-1)^{n-1} \pmod{2^{3n-1}}.$$

By induction on $n$, the power of 2 dividing $E^{j+2}(E^2-1)^{n-1}$ is equal to the power of 2 dividing $2^{n-1}(n-1)!$, and Frobenius's result follows.

In view of Theorem 9.2, it is natural to ask whether there are analogous congruences for generalized Euler numbers. There seem to be many possibilities, but the most attractive is given by the following conjecture: Define numbers $f_n^{(m)}$ by

$$\sum_{n=0}^{\infty} f_n^{(m)} \frac{x^{(2n+1)m}}{\big((2n+1)m\big)!} = \frac{x^m}{m!} \Big/ \sum_{n=0}^{\infty} \frac{x^{2nm}}{(2nm)!}.$$

(Thus $f_n^{(1)}$ is $F_{2n+1}$ as defined above.) Let $2^{\mu_{j,n,t}}$ be the highest power of 2 dividing

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f_{i+j}^{(2^t)}.$$

Then for $t \ge 1$, we have $\mu_{j,0,t} = 0$, $\mu_{j,1,t} = 4$, and for $n > 1$, $\mu_{j,n,t} = \lfloor \frac{7n}{2} \rfloor - 1$, except when $t = 1$, $n \equiv 2 \pmod 4$, and $n \ge 6$.

In the exceptional case, $\mu_{j,n,1} = \frac{7n}{2} + 2 + \rho_2(j + \vartheta_n)$, where $\vartheta_n$ is some integer or 2-adic integer. The first few values of $\vartheta_n$ (or reasonably good 2-adic approximations to them) are $\vartheta_6 = 118$, $\vartheta_{10} = 7$, $\vartheta_{14} = 2$, $\vartheta_{18} = 13$, $\vartheta_{22} = 32$, and $\vartheta_{26} = 27$.

By way of illustration, $\mu_{0,1,t} = 4$ for $t \ge 1$ is equivalent to the (easily proved) assertion that $1 + \binom{3 \cdot 2^t}{2^t}$ is divisible by 16 but not by 32.

## 10. Bell numbers

The Bell numbers $B_n$ are defined by the exponential generating function

$$B(x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = e^{e^x - 1}. \tag{10.1}$$

Although we are using the same notation for the Bell numbers that we used for Bernoulli numbers, there should be no confusion. Rota [34] proved several interesting properties of the Bell numbers using umbral calculus in his fundamental paper. Here we prove a well-known congruence of Touchard for Bell numbers and a generalization due to Carlitz.

Differentiating (10.1) gives $B'(x) = e^x B(x)$, so

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

With the Bell umbra $B$, given by $B^n = B_n$, this may be written $B^{n+1} = (B+1)^n$. Then by linearity, for any polynomial $f(x)$ we have

$$Bf(B) = f(B+1) \tag{10.2}$$

A consequence of (10.2), easily proved by induction, is that for any polynomial $f(x)$ and any nonnegative integer $n$,

$$B(B-1)\cdots(B-n+1)f(B)=f(B+n). \tag{10.3}$$

(A $q$-analogue of (10.3) has been given by Zeng [46, Lemma 8].) If we take $f(x)=1$ in (10.3), we obtain (since $B_0=1$)

$$B(B-1)(B-2)\cdots(B-n+1)=1. \tag{10.4}$$

We note that Rota took (10.4) as the *definition* of the Bell umbra and derived (10.2) and (10.1) from it.

As an application of these formulas, we shall prove Touchard's congruence for the Bell numbers [40, 41].

If $f(x)$ and $g(x)$ are two polynomials in $\mathbf{Z}[x]$, then by $f(x)\equiv g(x) \pmod p$, we mean that $f(x)-g(x)\in p\mathbf{Z}[x]$. We first recall two elementary facts about congruences for polynomials modulo a prime $p$. First we have Lagrange's congruence, $x(x-1)\cdots(x-p+1)\equiv x^p-x \pmod p$. Second, if $g(x)\in\mathbf{Z}[x]$ then $g(x+p)-g(x)\equiv 0 \pmod p$.

**Theorem 10.1.** *For any prime $p$ and any nonnegative integer $n$,*

$$B_{n+p}-B_{n+1}-B_n\equiv 0 \pmod p.$$

*Proof.* By Lagrange's congruence,

$$(B^p-B-1)B^n\equiv\big(B(B-1)\cdots(B-p+1)-1\big)B^n \pmod p.$$

By (10.3),

$$\big(B(B-1)\cdots(B-p+1)-1\big)B^n=(B+p)^n-B^n.$$

Since $(x+p)^n-x^n\equiv 0 \pmod p$, $p$ divides $(B+p)^n-B^n$. □

Next we prove a generalization of Touchard's congruence analogous to a Kummer congruence, due to Carlitz [6].

**Theorem 10.2.** *For any prime $p$ and any nonnegative integers $n$ and $k$,*

$$(B^p-B-1)^kB^n\equiv 0 \pmod{p^{\lceil k/2\rceil}}.$$

*Proof.* Let $L(x)$ be the polynomial $x(x-1)\cdots(x-p+1)-1$. First we show that it suffices to prove that for any polynomial $f(x)\in\mathbf{Z}[x]$, $L(B)^kf(B)\equiv 0 \pmod{p^{\lceil k/2\rceil}}$. To see this, note that we may write $L(x)=x^p-x-1-pR(x)$, where $R(x)\in\mathbf{Z}[x]$. Then $(B^p-B-1)^kB^n=(L(B)+pR(B))^kB^n=\sum_{i=0}^k\binom{k}{i}p^iL(B)^{k-i}R(B)^iB^n$, and our hypothesis will show that $p^iL(B)^{k-i}R(B)^iB^n$ is divisible by $p$ to the power $i+\lceil(k-i)/2\rceil=\lceil(k+i)/2\rceil\ge\lceil k/2\rceil$.

We now prove by induction on $k$ that for any polynomial $f(x)\in\mathbf{Z}[x]$,

$$L(B)^kf(B)\equiv 0 \pmod{p^{\lceil k/2\rceil}}.$$

The assertion is trivially true for $k=0$. For the induction step, note that we may write $L(x+p)=L(x)+pJ(x)$, where $J(x)\in\mathbf{Z}[x]$, and recall that by (10.3), $L(B)g(B)=$

$g(B+p)-g(B)$ for any polynomial $g$. Then for any $f(x) \in \mathbf{Z}[x]$ we have for $k>0$

$$\begin{aligned} L(B)^k f(B) &= L(B) \cdot L(B)^{k-1} f(B) = L(B+p)^{k-1} f(B+p) - L(B)^{k-1} f(B) \\ &= \big(L(B)+pJ(B)\big)^{k-1} f(B+p) - L(B)^{k-1} f(B) \\ &= L(B)^{k-1}\big(f(B+p)-f(B)\big) + \sum_{i=1}^{k-1} p^i \binom{k-1}{i} L(B)^{k-1-i} J(B)^i f(B+p). \end{aligned}$$

We show that each term of the last expression is divisible by $p^{\lceil k/2 \rceil}$. Since $f(x+p)-f(x) = ph(x)$ for some $h(x) \in \mathbf{Z}[x]$, we have $L(B)^{k-1}\big(f(B+p)-f(B)\big) = pL(B)^{k-1}h(B)$, which by induction is divisible by $p$ to the power $1+\lceil (k-1)/2 \rceil \geq \lceil k/2 \rceil$. By induction also, the $i$th term in the sum is divisible by $p$ to the power $i + \lceil (k-1-i)/2 \rceil = \lceil (k-1+i)/2 \rceil \geq \lceil k/2 \rceil$. This completes the proof. □

Theorem 10.2 can be extended in several ways (in particular, the modulus can be improved); see Lunnon, Pleasants, and Stephens [31] and Gessel [20], which both use umbral methods (though the latter is primarily combinatorial). Another congruence for Bell numbers, also proved umbrally, was given by Gertsch and Robert [18].

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