ON THE DECOMPOSITION MATRICES OF THE QUANTIZED SCHUR ALGEBRA

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Abstract. We prove the decomposition conjecture for the Schur algebra stated in [LT]. We also give a new approach to the Lusztig conjecture via canonical bases of the Hall algebra.

0. Introduction and general notations.

0.1. The aim of this paper is to give a proof of the decomposition conjecture for the quantized Schur algebra [LT, Conjecture 5.2] which generalizes the theorem of Ariki (see [A]) on the decomposition numbers of the Hecke algebra of type A. More precisely, let $\bigwedge^\infty$ be the level 1 Fock space of type A and let $\mathbf{B}^\pm$ be the bases of $\bigwedge^\infty$ introduced in [LT]. The decomposition conjecture links the decomposition matrices of the quantized Schur algebra and the basis $\mathbf{B}^+$. Our proof consists in two steps : first we express $\mathbf{B}^\pm$ in terms of some Kazhdan-Lusztig polynomials. Then we note that a simple module of the quantized Schur algebra can be pulled-back to a simple module of the Lusztig integral form of the quantized enveloping algebra of $\mathfrak{sl}_k$ (denoted by $\mathbf{U}(\mathfrak{sl}_k)$). Thus, the Lusztig conjecture for the dimension of the simple $\mathbf{U}(\mathfrak{sl}_k)$-modules at roots of unity identifies the entries of the decomposition matrices with some Kazhdan-Lusztig polynomials. It suffices to observe that these polynomials are precisely the ones which appear in $\mathbf{B}^+$.

Let $\mathbf{U}_n^-$ be the Hall algebra of nilpotent representations of the cyclic quiver. Set $\varepsilon = \exp(2i\pi/n')$. Put $n = n'$ if n odd, $n = n'/2$ else, i.e. n is the order of ε^2. Let $\mathbf{U}_\varepsilon(\mathfrak{sl}_k)$ be the specialization at $v = \varepsilon$ of $\mathbf{U}(\mathfrak{sl}_k)$. We give a new approach to the proof of the Lusztig conjecture on the character of the simple modules of $\mathbf{U}_\varepsilon(\mathfrak{sl}_k)$ in terms of the canonical basis of $\mathbf{U}_n^-$. Recall that this conjecture (proved by Kashiwara-Tanisaki and Kazhdan-Lusztig) gives the multiplicity of the Weyl module of $\mathbf{U}_\varepsilon(\mathfrak{sl}_k)$ with highest weight μ, say W_μ, in the simple $\mathbf{U}_\varepsilon(\mathfrak{sl}_k)$-module with highest weight λ, say V_λ, i.e.

$$(a) \qquad [V_\lambda : W_\mu] = \sum_y (-1)^{l(yx)} P_{yx}(1),$$

where $x \in \widehat{\mathfrak{S}}_k$ is minimal such that $\nu = \lambda \cdot x^{-1}$ satisfies

$$\nu_i < \nu_{i+1} \quad \forall i = 1, 2, ..., k-1, \qquad \nu_1 - \nu_k \geq 1 - k - n,$$

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and $\mu = \lambda \cdot x^{-1}y$. We proceed as follows. First we prove that $\bigwedge^\infty$ is a cyclic $\mathbf{U}_n^-$-module generated by the vacuum vector $|\emptyset\rangle$. Then we define a basis $\mathbf{B}'$ of $\mathbf{U}_n^-$ using intersection cohomology. We construct a basis $\mathbf{B}$ of $\bigwedge^\infty$ via the action of $\mathbf{B}'$ on the vacuum vector. We prove that $\mathbf{B}$ and $\mathbf{B}^+$ are fixed by the same semi-linear involution (see Theorem 6.3). At last, we prove that the equality $\mathbf{B} = \mathbf{B}^+$ is a q-analogue of the Lusztig conjecture (see Subsection 11.4). The reader should be warned that we endow the Hall algebra with the product opposit to the usual one (used in [G1] or [L1-4]).

The plan of the paper is the following. In Sections 1-4 we recall the definitions and the main properties of the basic objects. In Sections 5-6 we construct an action of $\mathbf{U}_n^-$ on the Fock space $\bigwedge^\infty$. Proposition 6.1 is new. In Section 7 we introduce the convolution algebra on pairs of affine flags. This algebra is a geometric analogue of the affine Schur algebra (Proposition 7.4) and is related to $\mathbf{U}_n^-$ in Proposition 7.6. In Sections 8-9 we give a representation of $\mathbf{U}_n^-$ on the finite wedges space, $\bigwedge^l$, via the coproduct of $\mathbf{U}_n^-$. This action is related to the convolution algebra on affine flags by Lemma 8.3. In Section 10 we interpret the action of $\mathbf{U}_n^-$ on $\bigwedge^\infty$ as a "limit" of $\bigwedge^l$ when l goes to infinity. Using the results of Sections 7-9 we prove that the elements of $\mathbf{B}$ are fixed by the Leclerc-Thibon involution (Theorem 6.3). In Section 11 we prove the Decomposition Conjecture. Let us observe that the proof only uses the results of Sections 8 and 9. In Section 12 we reinterpret the Lusztig conjecture. We use in an essentiel way the construction of the representation of $\mathbf{U}_n^-$ on $\bigwedge^\infty$ given in Section 6.

Contents.

0.2. We now fix a few general notations. Set $\mathbb{S} = \mathbb{C}[v]$, $\mathbb{A} = \mathbb{C}[v, v^{-1}]$. Let $\mathbb{F}$ be a field with q^2 elements and let $\bar{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. Fix a set I. For any $i \in I$ and $r \in \mathbb{N}^\times$, let $\bar{\mathbb{F}}^r[i]$ be the I-graded $\bar{\mathbb{F}}$-vector space with a single r-dimensional component, in degree i. Let $\epsilon_i \in \mathbb{N}^{(I)}$ be the dimension of $\bar{\mathbb{F}}[i]$. For any $d \in \mathbb{N}^{(I)}$ set $|d| = \sum_{i\in I} d_i$. If $i \in \mathbb{Z}$ let $\bar{\imath}$ be the class of i in $\mathbb{Z}/n\mathbb{Z}$. Given a positive integer l let $\Pi(l)$ be the set of all the partitions of l and let Π_l be the set of partitions with at most l parts. Put $\Pi = \cup_l \Pi(l)$. The set Π is endowed with the usual order. If $\lambda \in \Pi$ let λ' be the dual partition. For an irreducible algebraic variety X we denote by $\mathcal{H}^i(IC_X)$ the i-th cohomology sheaf of the intersection complex of X. Then, for any stratum $Y \subset X$, let $\dim \mathcal{H}^i_Y(IC_X)$ be the dimension of the stalk of $\mathcal{H}^i(IC_X)$ at a point of Y. For any set X with the action of a group G let $\mathbb{C}_G(X)$ be the set of G-invariant functions $X \to \mathbb{C}$ supported on a finite number of orbits. For any subset X of an algebraic variety let $\bar{X}$ denote its Zariski closure.

1. The Hecke algebra.

1.1. Fix $n \in \mathbb{N}^\times$ and set

$$A_l^n = \{\mathbf{i} \in \mathbb{Z}^l \,|\, 1-n \leq i_1 \leq i_2 \leq \cdots \leq i_l \leq 0\}.$$

Let $\mathfrak{S}_l$ be the symmetric group and let $\widehat{\mathfrak{S}}_l = \mathfrak{S}_l \ltimes \mathbb{Z}^l$ be the extended affine Weyl group. Let $\widehat{S}_l \subset \widehat{\mathfrak{S}}_l$ be the set of simple affine reflexions and put $S_l = \widehat{S}_l \cap \mathfrak{S}_l$. As usual, the simple affine reflexions are denoted by $s_0, s_1, ..., s_{l-1}$ in such a way that $S_l = \{s_1, s_2, ..., s_{l-1}\}$. Let $\pi \in \widehat{\mathfrak{S}}_l$ be the zero length element such that $s_{i-1} = \pi^{-1} s_i \pi$. The group $\widehat{\mathfrak{S}}_l$ acts on $\mathbb{Z}^l$ on the right in such a way that

$$(\mathbf{i})\lambda = \mathbf{i} + n\lambda \qquad \text{if} \quad \lambda \in \mathbb{Z}^l$$

$$(\mathbf{i})s_j = (i_1, i_2, ..., i_{j+1}, i_j, ..., i_l) \qquad \text{if} \quad j \neq 0$$

$$(\mathbf{i})s_0 = (i_l - n, i_2, ..., i_{l-1}, i_1 + n).$$

The alcove A_l^n is a fundamental domain for this action. If $\mathbf{i} \in A_l^n$ let $\mathfrak{S}_\mathbf{i} \subset \widehat{\mathfrak{S}}_l$ be its isotropy group, $S_\mathbf{i} = \widehat{S}_l \cap \mathfrak{S}_\mathbf{i}$, and let $\mathfrak{S}^\mathbf{i}$ be the set of minimal length representatives of the cosets in $\mathfrak{S}_\mathbf{i} \setminus \widehat{\mathfrak{S}}_l$. For any $x \in \widehat{\mathfrak{S}}_l$, let $x_\mathbf{i} \in \mathfrak{S}_\mathbf{i}$ and $x^\mathbf{i} \in \mathfrak{S}^\mathbf{i}$ be such that $x = x_\mathbf{i} x^\mathbf{i}$. Let $\omega \in \mathfrak{S}_l$ be the longest element. Set $\rho = (0, -1, -2, ..., 1-l) \in \mathbb{Z}^l$ and put

$$\lambda \cdot x = (\lambda + \rho)x - \rho, \qquad x \in \widehat{\mathfrak{S}}_l, \quad \forall \lambda \in \mathbb{Z}^l.$$

1.2. The Hecke algebra of type GL_l, say $\mathbf{H}_l$, is the unital associative $\mathbb{A}$-algebra generated by $T_i^{\pm 1}$, $i = 1, 2, ...l-1$ modulo the following relations

$$(a) \qquad \begin{aligned} &T_i\, T_i^{-1} = 1 = T_i^{-1}\, T_i, \qquad (T_i + 1)(T_i - v^{-2}) = 0, \\ &T_i\, T_{i+1}\, T_i = T_{i+1}\, T_i\, T_{i+1}, \qquad |i-j| > 1 \Rightarrow T_i\, T_j = T_j\, T_i. \end{aligned}$$

The affine Hecke algebra of type GL_l, say $\widehat{\mathbf{H}}_l$, is the unital associative $\mathbb{A}$-algebra generated by $T_i^{\pm 1}, X_j^{\pm 1}$, $i = 1, 2, ..., l-1, j = 1, 2, ..., l$ modulo the relations (a) and

$$X_i, X_i^{-1} = 1 = X_i^{-1}\, X_i, \qquad X_i\, X_j = X_j\, X_i,$$

$$T_i\, X_i\, T_i = v^{-2} X_{i+1}, \qquad j \neq i, i+1 \Rightarrow X_j\, T_i = T_i\, X_j.$$

For all $x \in \mathfrak{S}_l \ltimes \mathbb{Z}^l$ let $l(x)$ be the length of x and let $\tilde{T}_x$ be the normalized element $\tilde{T}_x = v^{l(x)} T_x$. The algebra $\widehat{\mathbf{H}}_l$ is isomorphic to the Hecke algebra of the extended affine Weyl group $\mathfrak{S}_l \ltimes \mathbb{Z}^l$ via the Bernstein isomorphism which maps $\tilde{T}_\lambda^{-1}$ to $X^\lambda = X_1^{\lambda_1} X_2^{\lambda_2} \cdots X_l^{\lambda_l}$ if $\lambda \in \mathbb{Z}^l$ is dominant, i.e. if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. The semilinear involution $^- : \widehat{\mathbf{H}}_l \to \widehat{\mathbf{H}}_l$ is such that $\bar{T}_x = T_{x^{-1}}^{-1}$ for all x. For all x put $\tilde{T}_x = v^{l(x)} T_x$. If $t \in \mathbb{C}^\times$ let $\widehat{\mathbf{H}}_{l|t}$ be the specialization of $\widehat{\mathbf{H}}_l$ at $v = t$.

2. The quantum group.

Put $I = \{1, 2, ..., n-1\}$ (resp. $I = \{0, 1, ..., n-1\}$) and let a_{ij} be the entries of the Cartan matrix of type A_{n-1} (resp. $A^{(1)}_{n-1}$). The quantized enveloping enveloping algebra of $\mathfrak{sl}_n$ (resp. $\widehat{\mathfrak{sl}}_n$) is the unital associative $\mathbb{C}(v)$-algebra generated by $\mathbf{e}_i, \mathbf{f}_i, \mathbf{k}_i^{\pm 1}$, $i \in I$, modulo the Kac-Moody type relations

$$\mathbf{k}_i\,\mathbf{k}_i^{-1} = 1 = \mathbf{k}_i^{-1}\,\mathbf{k}_i, \qquad \mathbf{k}_i\,\mathbf{k}_j = \mathbf{k}_j\,\mathbf{k}_i,$$

$$\mathbf{k}_i\,\mathbf{e}_j = v^{a_{ij}}\mathbf{e}_j\,\mathbf{k}_i, \quad \mathbf{k}_i\,\mathbf{f}_j = v^{-a_{ij}}\mathbf{f}_j\,\mathbf{k}_i, \quad [\mathbf{e}_i, \mathbf{f}_j] = \delta_{ij}\frac{\mathbf{k}_i - \mathbf{k}_i^{-1}}{v - v^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \mathbf{e}_i^{(k)}\,\mathbf{e}_j\,\mathbf{e}_i^{(1-a_{ij}-k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k \mathbf{f}_i^{(k)}\,\mathbf{f}_j\,\mathbf{f}_i^{(1-a_{ij}-k)} = 0 \quad \text{if} \quad i \neq j,$$

where

$$[k] = \frac{v^k - v^{-k}}{v - v^{-1}}, \qquad [k]! = [k]\,[k-1]\cdots[1], \qquad \mathbf{e}_i^{(k)} = \frac{\mathbf{e}_i^k}{[k]!}, \qquad \mathbf{f}_i^{(k)} = \frac{\mathbf{f}_i^k}{[k]!}.$$

We denote by $\mathbf{U}(\mathfrak{sl}_n)$ (resp. $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$) the Lusztig integral form, i.e. the $\mathbb{A}$-subalgebra generated by the divided powers $\mathbf{e}_i^{(k)}$, $\mathbf{f}_i^{(k)}$, and by $\mathbf{k}_i^{\pm 1}$. If $n = \infty$ the algebra $\mathbf{U}(\mathfrak{sl}_\infty)$ is well defined. The algebras above are Hopf algebras. The coproduct is

$$\Delta\mathbf{e}_i = \mathbf{e}_i \otimes \mathbf{k}_i + 1 \otimes \mathbf{e}_i, \qquad \Delta\mathbf{f}_i = \mathbf{f}_i \otimes 1 + \mathbf{k}_i^{-1} \otimes \mathbf{f}_i, \qquad \Delta\mathbf{k}_i = \mathbf{k}_i \otimes \mathbf{k}_i.$$

Let $\mathbf{U}^-(\widehat{\mathfrak{sl}}_n) \subset \mathbf{U}(\widehat{\mathfrak{sl}}_n)$ and $\mathbf{U}^-(\mathfrak{sl}_\infty) \subset \mathbf{U}(\mathfrak{sl}_\infty)$ be the subalgebras generated by the elements $\mathbf{f}_i^{(k)}$.

3. The Hall algebra.

In this section we recall some of the results of [L1-4] and [G1].

3.1. Fix a finite field $\mathbb{F}$ with q^2 elements as in the introduction. Let $\Gamma = (I, J)$ be an oriented graph : I is the set of vertices and J is the set of arrows. Given an arrow $j \in J$ let j_1 and j_2 be respectively the input vertex and the output vertex. Fix $d \in \mathbb{N}^{(I)}$ and let V be an I-graded $\mathbb{F}$-vector space of dimension d. Let $E_V \subseteq \bigoplus_{j \in J} \mathrm{Hom}\,(V_{j_1}, V_{j_2})$ be the subset of nilpotent representations of Γ on V. In this paper we will suppose that Γ is one of the following two graphs :

(a) $\Gamma = \Gamma_n$ is the cyclic quiver of type $A^{(1)}_n$, i.e. $I = \mathbb{Z}/n\mathbb{Z}$ and $J = \{\bar{\imath} \to \bar{\imath}+1 \,|\, \bar{\imath} \in \mathbb{Z}/n\mathbb{Z}\}$,

(b) $\Gamma = \Gamma_\infty$ is the infinite quiver of type A_∞, i.e. $I = \mathbb{Z}$ and $J = \{i \to i+1 \,|\, i \in \mathbb{Z}\}$.

3.2. Set $\mathbf{A}_d = \mathbb{C}_{G_V}(E_V)$ where $G_V = \prod_{i \in I} GL(V_i)$. Given $a, b \in \mathbb{N}^{(I)}$ such that $d = a + b$, fix I-graded $\mathbb{F}$-vector spaces U, W of dimensions a, b. Let consider the diagram

$$E_U \times E_W \xleftarrow{p_1} E \xrightarrow{p_2} F \xrightarrow{p_3} E_V,$$

where

(*c*) E is the set of triples (x,ϕ,ψ) such that $x\in E_V$,

$$0\to U\xrightarrow{\phi}V\xrightarrow{\psi}W\to 0$$

is an exact sequence of I-graded vector spaces and $\phi(U)$ is stable by x,

(*d*) F is the set of pairs (x,U') where $x\in E_V$ and $U'\subset V$ is a x-stable I-graded subspace of dimension a.

Given $f\in\mathbb{C}_{G_U}(E_U)$ and $g\in\mathbb{C}_{G_W}(E_W)$ set

$$f\circ g=q^{-m(b,a)}(p_3)_!h\in\mathbb{C}_{G_V}(E_V),$$

where $h\in\mathbb{C}(F)$ is the function such that $p_2^*h=p_1^*(fg)$ and $m(b,a)=\sum_{j\in J}b_{j_1}a_{j_2}+\sum_{i\in I}b_ia_i$. Then $(\mathbf{A},\circ)$, where $\mathbf{A}=\bigoplus_d\mathbf{A}_d$, is an associative algebra.

3.3. Given $a,b\in\mathbb{N}^{(J)}$ such that $d=a+b$, fix a I-graded $\mathbb{F}$-vector space $U\subset V$ of dimension a. Let consider the diagram

$$E_U\times E_{V/U}\xleftarrow{p}E\xrightarrow{i}E_V.$$

Here $E\subset E_V$ is the subset of representations preserving U, the map i is the inclusion and p is the obvious projection. Set

$$\Delta_{a,b}\,:\,\mathbf{A}_d\to\mathbf{A}_a\otimes\mathbf{A}_b,\quad f\mapsto q^{-n(b,a)}p_!i^*f,$$

where $n(b,a)=\sum_{j\in J}b_{j_1}a_{j_2}-\sum_{i\in I}b_ia_i$.

3.4. Recall that $\Gamma=\Gamma_n$ or Γ_∞. The classification of the isomorphism classes of nilpotent representations of Γ does not depend on the ground field $\mathbb{F}$. It is proved in [R] that the structural constants of $\mathbf{A}$ in the basis formed by the characteristic functions of the G_V-orbits in E_V are the value at $v=q$ of universal polynomials in $\mathbb{A}$. Thus $\mathbf{A}$ can be viewed as the specialization at $v=q$ of a $\mathbb{A}$-algebra, called the generic Hall algebra. Let $\mathbf{U}_n^-$ (resp. $\mathbf{U}_\infty^-$) be the generic Hall algebra if $\Gamma=\Gamma_n$ (resp. $\Gamma=\Gamma_\infty$). It is known that $\mathbf{U}_\infty^-$ is isomorphic to $\mathbf{U}^-(\mathfrak{sl}_\infty)$ and that $\mathbf{U}^-(\widehat{\mathfrak{sl}}_n)$ embeds in $\mathbf{U}_n^-$ (see [G1]). Let $\mathbf{A}^0$ be the $\mathbb{A}$-linear span of elements $\mathbf{k}_d$ with $d\in\mathbb{Z}^{(I)}$ such that

$$\mathbf{k}_0=1\quad\text{and}\quad\mathbf{k}_a\mathbf{k}_b=\mathbf{k}_{a+b},\quad\forall a,b.$$

For simplicity we will write $\mathbf{k}_i=\mathbf{k}_{\epsilon_i}$ for all $i\in I$. Set $\tilde{\mathbf{A}}=\mathbf{A}\otimes_{\mathbb{A}}\mathbf{A}^0$ and put

$$(f\otimes\mathbf{k}_a)\circ(g\otimes\mathbf{k}_b)=v^{-a\cdot d}(f\circ g)\otimes\mathbf{k}_{a+b},\qquad\forall g\in\mathbf{A}_d\quad\forall f\in\mathbf{A},$$

where $a\cdot d=-n(a,d)-n(d,a)$. Consider the map $\Delta\,:\,\tilde{\mathbf{A}}\to\tilde{\mathbf{A}}\otimes_{\mathbb{A}}\tilde{\mathbf{A}}$ such that

$$\Delta(f\otimes\mathbf{k}_c)=\sum_{d=a+b}\Delta_{a,b}(f)(\mathbf{k}_{b+c}\otimes\mathbf{k}_c),\quad\forall f\in\mathbf{A}_d.$$

Then $(\tilde{\mathbf{A}},\circ,\Delta)$ is a $\mathbb{A}$-bialgebra (it is due to Lusztig for the composition algebra, the general case is due to Green). Put $\tilde{\mathbf{U}}_n^-=\tilde{\mathbf{A}}$ if $\Gamma=\Gamma_n$ and $\tilde{\mathbf{U}}_\infty^-=\tilde{\mathbf{A}}$ if $\Gamma=\Gamma_\infty$.

3.5. Given a G_V-orbit $O \subset E_V$ let $\mathbf{f}_O \in \mathbf{A}$ be the $v^{\dim O}$ times the characteristic function of O. For any G_V-orbit $O \subset E_V$ set

$$\mathbf{b}_O = \sum_{i,O'} v^{-i+\dim O - \dim O'} \dim \mathcal{H}^i_{O'}(IC_O)\, \mathbf{f}_{O'}.$$

The elements $\mathbf{b}_O$ form a basis of $\mathbf{A}$. If $d \in \mathbb{N}^{(I)}$ let $\mathbf{f}_d \in \mathbf{A}$ be the characteristic function of the zero representation of Γ in a d-dimensional space. The following result is proved in Section 13.

Proposition. *The algebra* $\mathbf{A}$ *is generated by the* $\mathbf{f}_d$*,* $d \in \mathbb{N}^{(I)}$. □

3.6. Given two integers $i \le j$, let $\bar{\mathbb{F}}[i,j]$ be the unique indecomposable representation of Γ_∞ (resp. Γ_n) with dimension $\sum_{k=i}^{j} \epsilon_k$ (resp. $\sum_{k=i}^{j} \epsilon_{\bar{k}}$). For any partition $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots)$ let $\bar{\mathbb{F}}[\lambda]$ be the representation of Γ such that

$$\bar{\mathbb{F}}[\lambda] = \bigoplus_{k \ge 1} \bar{\mathbb{F}}[1-k, \lambda_k - k].$$

Let O_λ be the orbit of $\bar{\mathbb{F}}[\lambda]$ and put $d_\lambda = \dim O_\lambda$.

4. The Fock space.

In this section we recall the construction of the quantized Fock space, due to [H], as it is re-interpreted in [MM].

4.1. Let $T(\lambda)$ be the tableau of shape λ whose box with coordinates (x,y) is filled with $y - x$. For instance if $\lambda = (432)$ we get

$$\begin{array}{|c|c|c|c|} \hline -2 & -1 \\ \cline{1-3} -1 & 0 & 1 \\ \hline 0 & 1 & 2 & 3 \\ \hline \end{array}.$$

Let $\bigwedge^\infty$ be a $\mathbb{A}$-module with basis $\{|\lambda\rangle \mid \lambda \in \Pi\}$. If $i \in \mathbb{Z}$, a removable i-box of $T(\lambda)$ is a box with the color i which can be removed in such a way that the new tableau still comes from a partition. Similarly, an indent i-box corresponds to a box with the color i which can be added to $T(\lambda)$. Given $\bar{\imath} \in \mathbb{Z}/n\mathbb{Z}$, $i \in \bar{\imath}$, and a partition λ put

$$n_i(\lambda) = \sharp\{\text{indent } i\text{-box of } T(\lambda)\} - \sharp\{\text{removable } i\text{-box of } T(\lambda)\},$$

and $n_{\bar{\imath}}(\lambda) = \sum_{i \in \bar{\imath}} n_i(\lambda)$, $n_i^-(\lambda) = \sum_{j<i \& j \in \bar{\imath}} n_j(\lambda)$, $n_i^+(\lambda) = \sum_{j>i \& j \in \bar{\imath}} n_j(\lambda)$.

4.2. The algebra $\mathbf{U}(\mathfrak{sl}_\infty)$ acts on $\bigwedge^\infty$ by

$$\mathbf{k}_i(|\lambda\rangle) = v^{n_i(\lambda)}\, |\lambda\rangle, \quad \mathbf{e}_i(|\lambda\rangle) = |\nu\rangle, \quad \mathbf{f}_i(|\lambda\rangle) = |\mu\rangle,$$

where the partitions μ, ν are such that $T(\mu) - T(\lambda)$ and $T(\lambda) - T(\nu)$ are a box with color i. It is known that $\bigwedge^\infty$ is the simple module with highest weight Λ_0 (the fundamental weight) and that the canonical basis of $\bigwedge^\infty$ is $\{|\lambda\rangle \mid \lambda \in \Pi\}$. The weight multiplicities in $\bigwedge^\infty$ are 0 or 1, i.e. Λ_0 is a minuscule weight.

4.3. The algebra $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ acts on $\bigwedge^\infty$ by

$$\mathbf{k}_{\bar{\imath}}(|\lambda\rangle) = v^{n_{\bar{\imath}}(\lambda)}\,|\lambda\rangle, \quad \mathbf{e}_{\bar{\imath}}(|\lambda\rangle) = \sum_{i\in\bar{\imath}} v^{-n_i^-(\lambda)}\mathbf{e}_i(|\lambda\rangle), \quad \mathbf{f}_{\bar{\imath}}(|\lambda\rangle) = \sum_{i\in\bar{\imath}} v^{n_i^+(\lambda)}\mathbf{f}_i(|\lambda\rangle).$$

5. The representation of $\mathbf{U}_\infty^-$ on $\bigwedge^\infty$.

The algebras $\mathbf{U}_\infty^-$ and $\mathbf{U}^-(\mathfrak{sl}_\infty)$ are isomorphic. Thus $\bigwedge^\infty$ may be viewed as the quotient of $\mathbf{U}_\infty^-$ by a left ideal $\mathbf{I}$. Let us describe $\mathbf{I}$. Let $\bar{\Gamma}_\infty$ be the quiver Γ_∞ with the opposit orientation. For any $\mathbb{Z}$-graded $\bar{\mathbb{F}}$-vector space V let Λ_V be the variety of pairs $(x,\bar{x})$ of commuting representations respectively of Γ_∞ and $\bar{\Gamma}_\infty$ on V. The variety Λ_V is reducible. For any G_V-orbit $O \subset E_V$ set

$$\Lambda_O = \{(x,\bar{x}) \in \Lambda_V \,|\, x \in O\}.$$

According to [N] the orbit O is stable if there exists a triple

$$(x,\bar{x},i) \in \overline{\Lambda}_O \times \mathrm{Hom}(\bar{\mathbb{F}}[0], V)$$

such that i is homogeneous of degree 0 and that for any graded subspace $W \subseteq V$,

$$(a) \qquad (x(W), \bar{x}(W) \subseteq W \quad \text{and} \quad \mathrm{Im}\, i \subseteq W) \quad \Rightarrow \quad W = V$$

(since the Hall algebra is endowed with the product opposit to the usual one, we use the stability condition opposit to the one in [N]).

Proposition. *The ideal* $\mathbf{I}$ *is linearly spanned by the elements* $\mathbf{b}_O$ *such that* $O \neq O_\lambda$ *for all* λ. *Moreover the map* $\mathbf{U}_\infty^-/\mathbf{I} \to \bigwedge^\infty$, $\mathbf{b}_{O_\lambda} + \mathbf{I} \mapsto |\lambda\rangle$, *is an isomorphism of* $\mathbf{U}_\infty^-$*-modules.*

Proof. From [N, Theorem 11.7 and Proposition 3.5], $\mathbf{I}$ is linearly generated by the elements $\mathbf{b}_O$ such that O is unstable. Let us show that for any $\lambda \in \Pi$ the orbit O_λ is stable. A dimension counting then shows that the orbits O_λ are precisely all the stable orbits. Recall that $\bar{\mathbb{F}}[i,j]$ is the representation x of Γ_∞ on the graded space $\bigoplus_{k=i}^j \bar{\mathbb{F}}\, v_k$, where v_k is a non-zero vector of degree k, such that $x(v_k) = v_{k+1}$ if $k < j$ and $x(v_j) = 0$. Fix $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r) \in \Pi$. Fix non zero vectors $v_{k,s} \in \bar{\mathbb{F}}[1-k, \lambda_k - k]$ with degree s. The representation $\bar{\mathbb{F}}[\lambda]$ is given by the endomorphism x such that for all k,

$$x(v_{k,s}) = v_{k,s+1} \quad \text{if} \quad s \in [1-k, \lambda_k - k), \quad \text{and} \quad x(v_{k,\lambda_k - k}) = 0.$$

Let us exhibit a pair $(i,\bar{x})$ satisfying (a). Fix a graded homomorphism $i \in \mathrm{Hom}(\bar{\mathbb{F}}[0], \bar{\mathbb{F}}[\lambda])$ such that $v_{1,0} \in \mathrm{Im}\, i$. Consider the degree -1 linear operator $\bar{x}$ on $\bar{\mathbb{F}}[\lambda]$ such that

$$\bar{x}(v_{k,s}) = v_{k+1,s-1} \text{ if } k \neq r \text{ and } s \leq \lambda_{k+1} - k, \qquad \bar{x}(v_{k,s}) = 0 \text{ else.}$$

The operators x and $\bar{x}$ commute since

$$x\bar{x}(v_{r,s}) = 0 = \bar{x}x(v_{r,s}) \qquad \forall s,$$

$$x\bar{x}(v_{k,s}) = 0 = \bar{x}x(v_{k,s}) \qquad \forall s \geq \lambda_{k+1} - k,$$

$$x\bar{x}(v_{k,s}) = v_{k+1,s} = \bar{x}x(v_{k,s}) \qquad \forall s < \lambda_{k+1} - k,\ \forall k \neq r.$$

Now if $W \subseteq V$ is such that $x(W) \subseteq W$ and $\operatorname{Im} i \subseteq W$ then $\bar{\mathbb{F}}[0, \lambda_1 - 1] \subseteq W$: namely $v_{1,0} \in W$ and thus $v_{1,s} = x^s(v_{1,0}) \in W$ for all s. By definition of $\bar{x}$ we have for all $t < r$

$$\bar{x}^t(\bar{\mathbb{F}}[0, \lambda_1 - 1]) = \bar{\mathbb{F}}[-t, \lambda_{1+t} - 1].$$

The dimension d_λ of $\bar{\mathbb{F}}[\lambda]$ is such that $d_{\lambda,i}$ is the multiplicity of the color i in the tableau $T(\lambda)$ (see Section 4). Thus the linear isomorphism $\mathbf{b}_{O_\lambda} \mapsto |\lambda\rangle$ preserves the weights. Moreover it preserves the canonical base up to a permutation of its elements. Since Λ_0 is minuscule there is at most one vector of a given weight in the canonical basis. Hence the canonical bases are fully identified. □

6. The representation of $\mathbf{U}_n^-$ on $\bigwedge^\infty$.

6.1. Fix $d \in \mathbb{N}^{(\mathbb{Z})}$ and let V be a $\mathbb{Z}$-graded $\bar{\mathbb{F}}$-vector space of dimension d. Let $\bar{d} \in \mathbb{N}^{\mathbb{Z}/n\mathbb{Z}}$ be the multi-index such that $\bar{d}_{\bar{\imath}} = \sum_{j \in \bar{\imath}} d_j$ for all $\bar{\imath} \in \mathbb{Z}/n\mathbb{Z}$, and let $\bar{V}$ be the $\bar{d}$-dimensional $\mathbb{Z}/n\mathbb{Z}$-graded vector space such that $\bar{V}_{\bar{\imath}} = \bigoplus_{j \in \bar{\imath}} V_j$. The vector space $\bar{V}$ is filtered by the subspaces

$$\bar{V}_{\geq i} = \bigoplus_{j \geq i} V_j, \qquad \forall i \in \mathbb{Z}.$$

The associated graded is naturally identified with the $\mathbb{Z}$-graded space V. Set

$$E_{\bar{V},V} = \{x \in E_{\bar{V}} \,|\, x(\bar{V}_{\geq i}) \subseteq \bar{V}_{\geq i+1}, \quad \forall i\}.$$

The map $p : E_{\bar{V},V} \to E_V$ associate to a representation of Γ_n in $\bar{V}$ the corresponding graded representation of Γ_∞ in V. Let $j : E_{\bar{V},V} \hookrightarrow E_{\bar{V}}$ be the closed embedding. Let consider the map $\gamma_d : \mathbf{U}^-_{n,\bar{d}} \to \mathbf{U}^-_{\infty,d}$ such that

$$\gamma_{d|v=q^{-1}} : \mathbb{C}_{G_{\bar{V}}}(E_{\bar{V}}) \to \mathbb{C}_{G_V}(E_V), \quad f \mapsto q^{-h(d)} p_! j^*(f),$$

where $h(d) = \sum_{i<j \& \bar{\imath}=\bar{\jmath}} d_i(d_{j+1} - d_j)$. Put $k(b,a) = \sum_{i>j \& \bar{\imath}=\bar{\jmath}} b_i(2a_j - a_{j-1} - a_{j+1})$. The following is proved in Section 13.

Proposition. *Fix $\alpha, \beta \in \mathbb{N}^{\mathbb{Z}/n\mathbb{Z}}$ and $d \in \mathbb{N}^{(\mathbb{Z})}$ such that $\bar{d} = \alpha + \beta$. Then,*

$$\sum_{\substack{a+b=d \\ \bar{a}=\alpha, \bar{b}=\beta}} v^{-k(b,a)} \gamma_a(f) \circ \gamma_b(g) = \gamma_d(f \circ g) \qquad \forall f \in \mathbf{U}^-_{n,\alpha}, \forall g \in \mathbf{U}^-_{n,\beta},$$

□

Remark. With the notations in Section 3.5 we have $\gamma_d(\mathbf{f}_{\bar{d}}) = v^{h(d)} \mathbf{f}_d$. Observe that $\mathbf{f}_d$ is the product of the divided powers $\mathbf{f}_i^{(d_i)}$'s ordered from $i = -\infty$ to ∞.

6.2. For all $\lambda \in \Pi$ and all $x \in \mathbf{U}_n^-$ put

$$(a) \qquad x(|\lambda\rangle) = \sum_d \gamma_d(x) \mathbf{k}_{d'}(|\lambda\rangle) \quad \text{where} \quad d' = \sum_{j<i, \bar{\imath}=\bar{\jmath}} d_j \epsilon_i.$$

Corollary. *Formula (a) extends the Hayashi action of $\mathbf{U}^-(\widehat{\mathfrak{sl}}_n)$ on $\bigwedge^\infty$ to a representation of $\mathbf{U}_n^-$.*

Proof. The compatibility with the product in $\mathbf{U}_n^-$ follows from Proposition 6.1. Formula (a) implies that

$$\mathbf{f}_{\bar{\imath}}(|\lambda\rangle) = \sum_{i\in\bar{\imath}}\sum_{\mu} v^{g(\epsilon_i,d_\lambda)}|\mu\rangle,$$

where μ is a partition such that $T(\mu) - T(\lambda)$ is a box with color i and

$$g(\epsilon_i, d_\lambda) = -\sum_{\substack{i<j\\ \bar{\jmath}=\bar{\imath}}}(2d_{\lambda,j} - d_{\lambda,j-1} - d_{\lambda,j+1}) + \alpha_i,$$

where $\alpha_i = 1$ if $i < 0$ and $\bar{\imath} = 0$, and $\alpha_i = 0$ else. We have already observed that $d_{\lambda,i}$ is the multiplicity of the color i in $T(\lambda)$. Thus, $n_j(\lambda) = -2d_{\lambda,j}+d_{\lambda,j-1}+d_{\lambda,j+1}+\delta_{j0}$, and

$$g(\epsilon_i, d_\lambda) = \sum_{\substack{i<j\\ \bar{\jmath}=\bar{\imath}}} n_j(\lambda) = n_i^+(\lambda).$$

□

6.3. For any $\lambda \in \Pi$ set $\mathbf{b}_\lambda = \mathbf{b}_{O_\lambda}|\emptyset\rangle$ where O_λ is the isomorphism class of representations of Γ_n defined in Subsection 3.6. Put $\mathbf{B} = \{\mathbf{b}_\lambda \,|\, \lambda \in \Pi\}$. Leclerc and Thibon have introduced in [LT] a semi-linear involution on $\bigwedge^\infty$.

Theorem. $\mathbf{B}$ *is a basis of* $\bigwedge^\infty$ *whose elements are fixed by the Leclerc-Thibon involution.*

The theorem is proved in Subsection 10.1. We first introduce some more material.

6.4. Let $r : E_V \to E_{\bar{V}}$ be such that

$$r(x)_{|\bar{V}_{\bar{\imath}}} = \bigoplus\nolimits_{i\in\bar{\imath}} x_{|V_i} \qquad \forall x \in E_V.$$

Fix a $G_{\bar{V}}$-orbit $O \subset E_{\bar{V}}$ such that $O \cap E_{\bar{V},V} \neq \emptyset$. If $x \in p(O \cap E_{\bar{V},V})$ then

$$(b) \qquad \sharp(p^{-1}(x) \cap O) \in \begin{cases} q^{2\mathbb{N}} & \text{if} \quad r(x) \in O \\ (q^2-1)\mathbb{N} & \text{else.} \end{cases}$$

Indeed, fix $y \in p^{-1}(x)\cap O$. It suffices to consider the case where y is indecomposable. Then, fix a basis of homogeneous vectors $\{v_i \,|\, i \in [1, r]\}$ of $\bar{V}$ such that

$$(c) \qquad y(v_k) = v_{k+1} \qquad \forall k = 1, 2, ..., r-1.$$

If $r(x) \in \bar{O} \setminus O$ then there exist i, j, such that $v_i \in \bar{V}_{\geq j} \setminus \bar{V}_{>j}$ and $v_{i+1} \in \bar{V}_{\geq j}$. If $t \in \mathbb{F}^\times$ the representation $y_t \in E_{\bar{V},V}$ obtained by doing $v_k \mapsto tv_k$ for all $k \leq i$ in (c) is in $p^{-1}(x) \cap O$. Thus $\sharp(\mathbb{F}^\times) \,|\, \sharp(p^{-1}(x) \cap O)$. If $r(x) \in O$ then $r(x)$ and y are isomorphic since $r(x), y \in O$. Thus x is indecomposable and it is easy to see that $p^{-1}(x) \cap O$ is a vector space. We are done. The identity (b) implies the following lemma which is used in Section 12.

Lemma. *For any* $G_{\bar{V}}$*-orbit* $O \subseteq E_{\bar{V}}$ *we have*

$$\gamma_d(\mathbf{f}_O) = \mathbf{f}_{O'}\, mod\ (v-1)$$

where $O' \subseteq E_V$ is the unique G_V-orbit such that $r(O') \subseteq O$. □

7. Flag varieties.

7.1. Fix a positive integer l. Set $\mathbb{L} = \mathbb{F}((z))$ and $G = GL_l(\mathbb{L})$. A lattice in $\mathbb{L}^l$ is a free $\mathbb{F}[[z]]$-submodule of rank l. Let Y be set of sequences of lattices $L = (L_i)_{i\in\mathbb{Z}}$ such that

$$L_i \subseteq L_{i+1} \quad \text{and} \quad L_{i+n} = z^{-1} L_i.$$

The group G acts on Y in the obvious way. Let M be the set of all $\mathbb{Z}\times\mathbb{Z}$-matrices with non-negative entries, say $\mathbf{m} = (m_{ij})_{i,j\in\mathbb{Z}}$, such that $m_{i+n,j+n} = m_{ij}$. Set

$$M^l = \{\mathbf{m} \in M \mid \textstyle\sum_{i\in\mathbb{Z}} \sum_{j=1}^n m_{ij} = l\}.$$

The set M^l parametrizes the orbits of the diagonal action of G in $Y \times Y$: to $\mathbf{m}$ corresponds the set $Y_{\mathbf{m}}$ of the pairs (L', L) such that

$$m_{ij} = \dim_{\mathbb{F}}\left(\frac{L_{i+1} \cap L'_{j+1}}{(L_i \cap L'_{j+1}) + (L_{i+1} \cap L'_j)}\right).$$

For all $L \in Y$ let $Y_{\mathbf{m},L}$ be the fiber over L of the first projection $Y_{\mathbf{m}} \to Y$. If $Y_{\mathbf{m},L} \neq \emptyset$ then $Y_{\mathbf{m},L}$ is the set of $\mathbb{F}$-points of an algebraic variety whose dimension, denoted by $y(\mathbf{m})$, is independent of L. Let $\mathbf{1_m} \in \mathbb{C}_G(Y \times Y)$ be $q^{-y(\mathbf{m})}$ times the characteristic function of $Y_{\mathbf{m}}$. The convolution product, denoted $\star$, endows $\mathbb{C}_G(Y \times Y)$ with the structure of an associative algebra.

7.2. Let X be the set of sequences of lattices $L = (L_i)_{i\in\mathbb{Z}}$ such that

$$L_i \subseteq L_{i+1}, \quad L_{i+l} = z^{-1} L_i \quad \text{and} \quad \dim_{\mathbb{F}}(L_{i+1}/L_i) = 1.$$

The group G acts on X in the obvious way. The orbits of the diagonal action of G in $Y \times X$ are labelled by functions $\mathbf{i} : \mathbb{Z} \to \mathbb{Z}$ such that $\mathbf{i}(k + l) = \mathbf{i}(k) + n$ for all k : let $X_{\mathbf{i}}$ be the orbit of the pair $(L_{\mathbf{i}}, L_\emptyset)$ such that

$$L_{\mathbf{i},i} = \prod_{\mathbf{i}(j)\leq i} \mathbb{F}\, e_j \quad \text{and} \quad L_{\emptyset,i} = \prod_{j\leq i} \mathbb{F}\, e_j.$$

Here $(e_1, e_2, ..., e_l)$ is a fixed $\mathbb{L}$-basis of $\mathbb{L}^l$ and $e_{i+lk} = z^{-k} e_i$ for all $k \in \mathbb{Z}$. A periodic function $\mathbf{i}$ as above is identified with the l-uple $(\mathbf{i}(1), \mathbf{i}(2), ..., \mathbf{i}(l)) \in \mathbb{Z}^l$. If $L \in Y$ let $X_{\mathbf{i},L}$ be the fiber over L of the projection $X_{\mathbf{i}} \to Y$. If $X_{\mathbf{i},L} \neq \emptyset$, then $X_{\mathbf{i},L}$ is the set of $\mathbb{F}$-points of an algebraic variety of dimension $l(\omega_{\mathbf{i}})$. Let $\mathbf{1_i} \in \mathbb{C}_G(Y \times X)$ be $q^{-l(\omega_{\mathbf{i}})}$ times the characteristic function of $X_{\mathbf{i}}$. The space $\mathbb{C}_G(Y \times X)$ is a left $\mathbb{C}_G(Y \times Y)$-module and a right $\mathbb{C}_G(X \times X)$-module.

7.3. For all $x \in \widehat{\mathfrak{S}}_l$ let $X_x \subset X \times X$ be the G-orbit of the pair $(x(L_\emptyset), L_\emptyset)$. There is an algebras isomorphism $\widehat{\mathbf{H}}_{l|q^{-1}} \tilde{\to} \mathbb{C}_G(X \times X)$ which maps T_x to the characteristic function of X_x (see [IM]). Put $P = X_1^{-1}\tilde{T}_1^{-1}\tilde{T}_2^{-1} \cdots \tilde{T}_{l-1}^{-1}$.

Lemma. *The right action of* $\widehat{\mathbf{H}}_l$ *on* $\mathbb{C}_G(Y \times X)$ *is such that if* $\mathbf{i} \in A_l^n$, $x \in \mathfrak{S}^{\mathbf{i}}$, *and* $s \in \widehat{S}_l$, *then* $(\mathbf{1}_{\mathbf{i}})\, P = \otimes \mathbf{1}_{(\mathbf{i})\pi}$ *and*

$$(\mathbf{1}_{(\mathbf{i})x})\, \tilde{T}_s = \begin{cases} v^{-1}\mathbf{1}_{(\mathbf{i})x} & \text{if} \quad xs \notin \mathfrak{S}^{\mathbf{i}} \quad (\text{then} \quad xs > x), \\ \mathbf{1}_{(\mathbf{i})xs} & \text{if} \quad xs > x \quad \text{and} \quad xs \in \mathfrak{S}^{\mathbf{i}}, \\ \mathbf{1}_{(\mathbf{i})xs} + (v^{-1} - v)\mathbf{1}_{(\mathbf{i})x} & \text{if} \quad xs < x \quad (\text{then} \quad xs \in \mathfrak{S}^{\mathbf{i}}). \end{cases}$$

Proof. To simplify the notations fix $l = 2$. Fix $\mathbf{i} \in A_l^n$ and $x \in \mathfrak{S}^{\mathbf{i}}$. Set $(i, j) = (\mathbf{i})x$. Then

$$xs_1 \notin \mathfrak{S}^{\mathbf{i}} \iff (\exists t \in S_{\mathbf{i}} \quad \text{such that} \quad xs_1 = tx) \iff (i, j)s_1 = (i, j).$$

Moreover,

$$xs_1 > x \quad \text{and} \quad xs_1 \in \mathfrak{S}^{\mathbf{i}} \iff i \leq j \quad \text{and} \quad (i, j)s_1 \neq (i, j).$$

The formulas in the proposition gives

$$(\mathbf{1}_{(i,j)})\, T_1 = \begin{cases} v^{-2}\mathbf{1}_{(i,j)} & \text{if} \quad i = j, \\ v^{-1}\mathbf{1}_{(j,i)} & \text{if} \quad i < j, \\ v^{-1}\mathbf{1}_{(j,i)} + (v^{-2} - 1)\mathbf{1}_{(i,j)} & \text{if} \quad i > j. \end{cases}$$

These are precisely the formulas in [VV, Section 5] taking into account the different normalizations for the Hecke algebra and the factor $q^{-l(\omega_{\mathbf{i}})}$. The result follows from [VV, Proposition 6]. □

7.4. Fix $\mathbf{i} \in A_l^n$. Let $\mathbf{H}_{\mathbf{i}} \subseteq \widehat{\mathbf{H}}_l$ be the parabolic subalgebra associated to $\mathfrak{S}_{\mathbf{i}}$. Set $e_{\mathbf{i}} = \sum_{x \in \mathfrak{S}_{\mathbf{i}}} T_x$ and $\pi_{\mathbf{i}} = \sum_{x \in \mathfrak{S}_{\mathbf{i}}} v^{-2l(x)}$. Thus $e_{\mathbf{i}}^2 = \pi_{\mathbf{i}} e_{\mathbf{i}}$ and $\bar{e}_{\mathbf{i}} = v^{2l(\omega_{\mathbf{i}})} e_{\mathbf{i}}$. Set

$$\mathbf{T}_{n,l} = \bigoplus_{\mathbf{i} \in A_l^n} e_{\mathbf{i}}\, \widehat{\mathbf{H}}_l.$$

The affine q-Schur algebra $\widehat{\mathbf{S}}_{n,l}$, introduced in [G2], is the endomorphism ring of the right $\widehat{\mathbf{H}}_l$-module $\mathbf{T}_{n,l}$. If $\mathbf{j} \in A_l^n$ set

$$M_{\mathbf{ij}} = \{\mathbf{m} \in M^l \,|\, Y_{\mathbf{m}} \cap (G(L_{\mathbf{i}}) \times G(L_{\mathbf{j}})) \neq \emptyset\}.$$

A matrix $\mathbf{m} \in \mathbf{M}_{\mathbf{ij}}$ is identified with the class in $\mathfrak{S}_{\mathbf{i}} \setminus \widehat{\mathfrak{S}}_l / \mathfrak{S}_{\mathbf{j}}$ of the elements x such that $(L_{(\mathbf{i})x}, L_{\mathbf{j}}) \in Y_{\mathbf{m}}$. Set $T_{\mathbf{m}} = \sum_{x \in \mathbf{m}} T_x$. Let $\widehat{\mathbf{H}}_{\mathbf{ij}} \subseteq \widehat{\mathbf{H}}_l$ be the $\mathbb{A}$-linear span of the elements $T_{\mathbf{m}}$ with $\mathbf{m} \in M_{\mathbf{ij}}$. The $\mathbb{A}$-linear homomorphism

$$\bigoplus_{\mathbf{i},\mathbf{j} \in A_l^n} \widehat{\mathbf{H}}_{\mathbf{ij}} \to \widehat{\mathbf{S}}_{n,l}$$

which maps $T_{\mathbf{m}}$, $\mathbf{m} \in M_{\mathbf{ij}}$, to the endomorphism such that $e_{\mathbf{k}} \mapsto \delta_{\mathbf{k},\mathbf{j}}\, T_{\mathbf{m}} \in e_{\mathbf{i}}\widehat{\mathbf{H}}_l$, is invertible. The product in the affine q-Schur algebra, denoted $\bullet$, is

$$T_{\mathbf{m}} \bullet T_{\mathbf{n}} = \delta_{\mathbf{k},\mathbf{j}}\, \pi_{\mathbf{j}}^{-1} T_{\mathbf{m}} T_{\mathbf{n}} \qquad \forall \mathbf{m} \in M_{\mathbf{ij}} \quad \forall \mathbf{n} \in M_{\mathbf{kl}}.$$

If $t \in \mathbb{C}^\times$ let $\widehat{\mathbf{S}}_{n,l|t}$ and $\mathbf{T}_{n,l|t}$ be the specializations of $\widehat{\mathbf{S}}_{n,l}$ and $\mathbf{T}_{n,l}$ at $v = t$.

Proposition. *(a) The map $\Phi : \widehat{\mathbf{S}}_{n,l|q^{-1}} \to \mathbb{C}_G(Y \times Y)$, $T_{\mathbf{m}} \mapsto q^{y(\mathbf{m})} \mathbf{1}_{\mathbf{m}}$, is an isomorphism of algebras.*
(b) There is a unique isomorphism of $\widehat{\mathbf{S}}_{n,l|q^{-1}} \times \widehat{\mathbf{H}}_{l|q^{-1}}$-modules, still denoted Φ, from $\mathbf{T}_{n,l|q^{-1}}$ to $\mathbb{C}_G(Y \times X)$ such that $e_{\mathbf{i}} \mapsto q^{l(\omega_{\mathbf{i}})} \mathbf{1}_{\mathbf{i}}$ for all $\mathbf{i} \in A_l^n$.

Proof. The map Φ is a linear isomorphism. For all $x, y, z \in \widehat{\mathfrak{S}}_l$ let $B_{xy}^z(v) \in \mathbb{A}$ be such that

$$T_x T_y = \textstyle\sum_z B_{xy}^z(v) T_z.$$

If $(L'', L) \in X_z$ then,

$$B_{xy}^z(q^{-1}) = \sharp\{L' \in X \,|\, (L'', L') \in X_x \quad \& \quad (L', L) \in X_y\}.$$

Fix $\mathbf{m} \in M_{\mathbf{ik}}$, $\mathbf{n} \in M_{\mathbf{ij}}$, and $\mathbf{o} \in M_{\mathbf{jk}}$. Let $A_{\mathbf{no}}^{\mathbf{m}} \in \mathbb{N}$ be such that

$$\mathbf{1}_{\mathbf{n}} \star \mathbf{1}_{\mathbf{o}} = \sum_{\mathbf{m}} q^{-y(\mathbf{n})-y(\mathbf{o})+y(\mathbf{m})} A_{\mathbf{no}}^{\mathbf{m}}\, \mathbf{1}_{\mathbf{m}}.$$

Then for any $z \in \mathbf{m}$,

$$A_{\mathbf{no}}^{\mathbf{m}} = h_{\mathbf{j}}^{-1} \textstyle\sum_{\substack{x \in \mathbf{n} \\ y \in \mathbf{o}}} B_{xy}^z(q^{-1}),$$

where $h_{\mathbf{j}} = \pi_{\mathbf{j}|v=q^{-1}}$ is the cardinal of the fiber of the projection $X \to G(L_{\mathbf{j}})$. Claim (a) follows from the identity

$$T_{\mathbf{n}} \bullet T_{\mathbf{o}} = \pi_{\mathbf{j}}^{-1} \textstyle\sum_z \sum_{\substack{x \in \mathbf{n} \\ y \in \mathbf{o}}} B_{xy}^z(v) T_z = \sum_{\mathbf{m}} A_{\mathbf{no}}^{\mathbf{m}} T_{\mathbf{m}} \mod (v - q^{-1}).$$

Let Φ be the unique isomorphism of right $\widehat{\mathbf{H}}_l$-modules $\mathbf{T}_{n,l|q^{-1}} \xrightarrow{\sim} \mathbb{C}_G(Y \times X)$ such that $\Phi(e_{\mathbf{i}}) = q^{l(\omega_{\mathbf{i}})} \mathbf{1}_{\mathbf{i}}$ for all $\mathbf{i} \in A_l^n$. Let us prove that Φ commutes to the action of $\widehat{\mathbf{S}}_{n,l}$. We must prove that for all $\mathbf{i}, \mathbf{j} \in A_l^n$ and all $\mathbf{m} \in M_{\mathbf{ij}}$ then

$$q^{y(\mathbf{m})} \mathbf{1}_{\mathbf{m}} \star \mathbf{1}_{\mathbf{j}} = q^{-l(\omega_{\mathbf{j}})} h_{\mathbf{i}}^{-1} \Phi(e_{\mathbf{i}} T_{\mathbf{m}}) = q^{l(\omega_{\mathbf{i}})-l(\omega_{\mathbf{j}})} h_{\mathbf{i}}^{-1} (\mathbf{1}_{\mathbf{i}})\, T_{\mathbf{m}}.$$

Put

$$\mathbf{1}_{\mathbf{m}} \star \mathbf{1}_{\mathbf{j}} = \sum_{\mathbf{k} \in \mathbb{Z}^l} q^{-y(\mathbf{m})-l(\omega_{\mathbf{j}})+l(\omega_{\mathbf{k}})} A_{\mathbf{mj}}^{\mathbf{k}} \mathbf{1}_{\mathbf{k}}, \qquad (\mathbf{1}_{\mathbf{i}}) T_{\mathbf{m}} = \sum_{\mathbf{k} \in \mathbb{Z}^l} q^{-l(\omega_{\mathbf{i}})+l(\omega_{\mathbf{k}})} A_{\mathbf{im}}^{\mathbf{k}} \mathbf{1}_{\mathbf{k}}.$$

Fix $z \in \widehat{\mathfrak{S}}_l$ and $(L'', L) \in X_z$ whose projection in $Y \times X$ is in $X_{\mathbf{k}}$. Then

$$A_{\mathbf{mj}}^{\mathbf{k}} = \sharp\{L' \in Y \,|\, (L'', L') \in Y_{\mathbf{m}}, \quad (L', L) \in X_{\mathbf{j}}\} = h_{\mathbf{j}}^{-1} \sum_{\substack{y \in \mathbf{m} \\ x \in \mathfrak{S}_{\mathbf{j}}}} B_{yx}^z(q^{-1}),$$

$$A^{\mathbf{k}}_{\mathbf{im}} = \sum_{y\in\mathbf{m}} \sharp\{L' \in X \,|\, (L'', L') \in X_{\mathbf{i}}, \quad (L', L) \in X_y\} = \sum_{\substack{y\in\mathbf{m}\\ x\in\mathfrak{S}_{\mathbf{i}}}} B^z_{xy}(q^{-1}).$$

Claim (b) follows from the equality

$$\pi_{\mathbf{i}}^{-1} \sum_{\substack{y\in\mathbf{m}\\ x\in\mathfrak{S}_{\mathbf{i}}}} T_x T_y = \pi_{\mathbf{j}}^{-1} \sum_{\substack{y\in\mathbf{m}\\ x\in\mathfrak{S}_{\mathbf{j}}}} T_y T_x.$$

□

7.5. The set $M^+ = \{\mathbf{m} \in M \,|\, i > j \Rightarrow m_{ij} = 0\}$ parametrizes the isomorphism classes of nilpotent representations of the quiver Γ_n : $O_{\mathbf{m}}$ is the class of $\bigoplus_{i=1}^n \bigoplus_{j\geq i} \bar{\mathbb{F}}[i,j]^{m_{ij}}$. Let $^-$ be the unique semilinear involution on $\mathbf{U}_n^-$ fixing the elements $\mathbf{b}_{O_{\mathbf{m}}}$.

Proposition. *The involution* $^-$ *on* $\mathbf{U}_n^-$ *is a ring homomorphism and* $\bar{\mathbf{f}}_\alpha = \mathbf{f}_\alpha$ *for all* $\alpha \in \mathbb{N}^{(\mathbb{Z}/n\mathbb{Z})}$.

Proof. The second claim is obvious since $\mathbf{f}_\alpha$ is the characteristic function of a single point. We now prove the first claim. For any algebraic variety X over $\bar{\mathbb{F}}$ let $\mathcal{D}(X)$ be the bounded derived category of complexes of $\mathbb{Q}_l$-sheaves on X (see [L2], [L3]). If G is a connected algebraic group acting on X let $\mathcal{D}^{ss}_G(X)$ be the full subcategory whose objects are sums of shifted simple G-equivariant objects in $\mathcal{D}(X)$. Lusztig has defined in [L2, Section 3.1] a convolution product

$$* \,:\, \mathcal{D}^{ss}_{G_U}(E_U) \times \mathcal{D}^{ss}_{G_W}(E_W) \to \mathcal{D}^{ss}_{G_V}(E_V)$$

such that $\mathcal{F} * \mathcal{G} = (p_3)_! \mathcal{H}$ where $\mathcal{H}$ satisfies $p_2^* \mathcal{H} \simeq p_1^*(\mathcal{F} \otimes \mathcal{G})$. Let D be the Verdier duality. Since p_1 and p_2 are smooth with connected fibers and since p_3 is proper we get $D(\mathcal{F} * \mathcal{G}) = (D\mathcal{F}) * (D\mathcal{G})[2d_1 - 2d_2]$ where d_1 and d_2 are the dimensions of the fibers of p_1 and p_2. Let α, β be the dimension of U, W. We know that $d_2 = \sum_{\bar{\imath}} \alpha_{\bar{\imath}}^2 + \sum_{\bar{\imath}} \beta_{\bar{\imath}}^2$ and $d_1 = d_2 + m(\beta, \alpha)$. Thus $D(\mathcal{F} * \mathcal{G}) = (D\mathcal{F}) * (D\mathcal{G})[2m(\beta, \alpha)]$. Finally observe that the elements $\mathbf{b}_{O_{\mathbf{m}}}$ are the Frobenius traces of the simple perverse sheaves on the E_V since the varieties $\bar{O}_{\mathbf{m}}$ are pure (see [L1, Corollary 11.6]). □

7.6. If $L', L \in Y$ are such that $L' \subseteq L$ then L/L' may be viewed as a nilpotent representation of Γ_n of dimension α where $\alpha_{\bar{\imath}} = \dim_{\mathbb{F}}(L_i/L'_i)$ (see [L2, Section 11], [GV]). Then, set

$$a(L', L) = \sum_{i=1}^n \dim_{\mathbb{F}}(L_i/L'_i)(\dim_{\mathbb{F}}(L'_{i+1}/L'_i) - \dim_{\mathbb{F}}(L_i/L'_i)).$$

Let $\Theta \,:\, \mathbf{U}_n^- \to \widehat{\mathbf{S}}_{n,l}$ be the $\mathbb{A}$-linear map such that

$$\Phi \circ \Theta(f)(L', L) = q^{-a(L',L)} f(L/L') \quad \text{if} \quad L' \subseteq L, \quad 0 \quad \text{else.}$$

If $\mathbf{i} \in A_l^n$ and $\mathbf{m} \in M^+$ let $\mathbf{m}^{\mathbf{i}} \in \cup_{\mathbf{j}} M_{\mathbf{ij}}$ be the matrix with the (i,j)-th entry equal to

$$\delta_{ij}(\sharp\mathbf{i}^{-1}(j+1) - \textstyle\sum_{k\leq j} m_{kj}) + (1 - \delta_{ij}) m_{i+1,j}.$$

Let ϕ be the semilinear involution on $\widehat{\mathbf{S}}_{n,l}$ such that $\phi(u) = v^{-2l(\omega_{\mathbf{j}})}\bar{u}$ for all $u \in \widehat{\mathbf{H}}_{\mathbf{ij}}$.

Proposition. *The map* $\Theta : \mathbf{U}_n^- \to \widehat{\mathbf{S}}_{n,l}$ *is an algebra homomorphism. Moreover if* $u \in \mathbf{U}_n^-$ *and* $\mathbf{m} \in M^+$ *we have*

$$\phi \circ \Theta(u) = \Theta(\bar{u}) \quad \textit{and} \quad \Phi \circ \Theta(\mathbf{f}_{O_\mathbf{m}}) = \sum_{\mathbf{i}\,|\,\mathbf{m}^\mathbf{i} \in M} \mathbf{1}_{\mathbf{m}^\mathbf{i}}.$$

Proof. The first claim is immediate from the formula

$$a(L'', L) - a(L', L) - a(L'', L') = -m(L/L', L'/L'')$$

and from the definition of the product in $\mathbf{U}_n^-$ and $\mathbb{C}_G(Y \times Y)$. We know that $\bar{\mathbf{f}}_\alpha = \mathbf{f}_\alpha$ for all $\alpha \in \mathbb{N}^{(\mathbb{Z}/n\mathbb{Z})}$. Similarly, $\phi \circ \Theta(\mathbf{f}_\alpha) = \Theta(\mathbf{f}_\alpha)$ since for any flag L' the L's such that $L' \subseteq L$ and $\mathbf{f}_\alpha(L/L') \neq 0$ are the rational points of a smooth variety. Hence, the second claim results from the first claim, Proposition 3.5, and the fact that ϕ is a ring homomorphism. Now let us first prove that

$$(c) \qquad (L', L) \in Y_{\mathbf{m}^\mathbf{i}} \quad \iff \quad \Big(L/L' \in O_\mathbf{m} \quad \text{and} \quad \dim_\mathbb{F}(L'_i/L'_{i-1}) = \sharp\mathbf{i}^{-1}(i) \Big).$$

By definition $(L', L) \in Y_{\mathbf{m}^\mathbf{i}}$ if and only if

$$\delta_{ij}(\sharp\mathbf{i}^{-1}(j+1) - \textstyle\sum_{k \leq j} m_{kj}) + (1 - \delta_{ij}) m_{i+1,j} = \dim_\mathbb{F}\left(\frac{L_{i+1} \cap L'_{j+1}}{(L_i \cap L'_{j+1}) + (L_{i+1} \cap L'_j)} \right).$$

Thus it suffices to prove that if $\dim_\mathbb{F}(L'_{i+1}/L'_i) = \sharp\mathbf{i}^{-1}(i+1)$ and $L' \subseteq L$ then

(d) $\dim_\mathbb{F}\left(\frac{L_i \cap L'_{j+1}}{(L_{i-1} \cap L'_{j+1}) + (L_i \cap L'_j)} \right)$ is the multiplicity of $\mathbb{F}[i, j]$ in L/L' for all $i \leq j$,

(e) if $x_{\bar{\imath}} : L_i/L'_i \to L_{i+1}/L'_{i+1}$ is the map induced by the inclusion $L_i \subseteq L_{i+1}$, then

$$\sharp\mathbf{i}^{-1}(i+1) - \dim_\mathbb{F} \operatorname{Ker}(x_{\bar{\imath}}) = \dim(L'_{i+1}/(L_i \cap L'_{i+1})).$$

Claim (e) is immediate since

$$\operatorname{Ker}(x_{\bar{\imath}}) = (L_i \cap L'_{i+1})/L'_i, \qquad L'_{i+1}/(L_i \cap L'_{i+1}) \simeq \frac{L'_{i+1}/L'_i}{(L_i \cap L'_{i+1})/L'_i},$$

and since $\sharp\mathbf{i}^{-1}(i+1) = \dim(L'_{i+1}/L'_i)$. Part (d) is due to the fact that $\mathbb{F}[i, j]$ is a direct summand of L/L' if and only if there is a vector $w \in L'_{j+1} \setminus L'_j$ such that $w \in L_i \setminus L_{i-1}$. The second claim follows from (c) and the formula

$$(\mathbf{m} \in M^+ \quad \text{and} \quad \mathbf{m}^\mathbf{i} \in M) \Rightarrow \dim O_\mathbf{m} + a(\mathbf{m}^\mathbf{i}) = y(\mathbf{m}^\mathbf{i}),$$

which is left to the reader. □

Remark. For any $\mathbf{m} \in M^l$ set

$$\mathbf{c}_\mathbf{m} = \sum_{i,\mathbf{n}} v^{-i+y(\mathbf{m})} \dim \mathcal{H}^i_{Y_{\mathbf{n},L}}(IC_{Y_{\mathbf{m},L}})\, T_\mathbf{n},$$

where L is such that $Y_{\mathbf{m},L} \neq \emptyset$. The elements $\mathbf{c_m}$ form a $\mathbb{A}$-basis of $\widehat{\mathbf{S}}_{n,l}$ and $\phi(\mathbf{c_m}) = \mathbf{c_m}$. Proposition 7.6 implies that for any $\mathbf{m} \in M^+$ we have

$$\Theta(\mathbf{b}_{O_\mathbf{m}}) = \sum_{\mathbf{i}\,|\,\mathbf{m^i}\in M} \mathbf{c_{m^i}}.$$

We will not use this.

8. The tensor representation of $\tilde{\mathbf{U}}_n^-$.

8.1. Let $\mathbb{A}^{(\mathbb{Z})}$ be the $\mathbb{A}$-linear span of vectors x_i, $i \in \mathbb{Z}$. Let $\mathbf{e}_{ij} \in M$ be the matrix with 1 at the spot (k,l) if $(k,l) \in (i,j) + \mathbb{Z}\,(n,n)$ and 0 elsewhere.

Lemma. *$\tilde{\mathbf{U}}_n^-$ acts on $\mathbb{A}^{(\mathbb{Z})}$ in such a way that for all $\mathbf{m} \in M$ and all $\alpha \in \mathbb{N}^{\mathbb{Z}/n\mathbb{Z}}$,*

$$\mathbf{f}_{O_\mathbf{m}}(\mathrm{x}_i) = \sum_{j\geq i} \delta_{\mathbf{m},\mathbf{e}_{ij}}\, \mathrm{x}_{j+1} \qquad \textit{and} \qquad \mathbf{k}_\alpha(\mathrm{x}_i) = v^{-n(\alpha,\epsilon_{\bar\imath})}\mathrm{x}_i.$$

Proof. It is the action obtained by taking $l=1$ in the geometric construction of Section 7 via the isomorphism $\mathbf{T}_{n,1} \tilde{\to} \mathbb{A}^{(\mathbb{Z})}$, $\mathbf{1}_i \mapsto \mathrm{x}_i$ (if $l=1$ then X and Y are zero dimensional). □

8.2. Put $\bigotimes^l = (\mathbb{A}^{(\mathbb{Z})})^{\otimes l}$. For any sequence $\mathbf{i} = (i_1, i_2, ..., i_l) \in \mathbb{Z}^l$ set $\otimes \mathrm{x}_\mathbf{i} = \mathrm{x}_{i_1} \otimes \mathrm{x}_{i_2} \otimes \cdots \otimes \mathrm{x}_{i_l}$. On one hand $\bigotimes^l$ is a left $\tilde{\mathbf{U}}_n^-$-module via the coproduct Δ. On the other hand $\widehat{\mathbf{H}}_l$ acts on $\bigotimes^l$ as follows for all $k = 1, 2, ..., l-1$ and $j = 1, 2, ..., l$:

$$(a) \qquad (\otimes \mathrm{x}_\mathbf{i})T_k = \begin{cases} v^{-2} \otimes \mathrm{x}_\mathbf{i} & \text{if} \quad i_k = i_{k+1} \\ v^{-1} \otimes \mathrm{x}_{(\mathbf{i})s_k} & \text{if} \quad -n < i_k < i_{k+1} \leq 0 \\ v^{-1} \otimes \mathrm{x}_{(\mathbf{i})s_k} + (v^{-2}-1) \otimes \mathrm{x}_\mathbf{i} & \text{if} \quad -n < i_{k+1} < i_k \leq 0, \end{cases}$$

$$(b) \qquad (\otimes \mathrm{x}_\mathbf{i})X_j^{-1} = \otimes \mathrm{x}_{(\mathbf{i})\epsilon_j}.$$

Lemma. *The representations of $\tilde{\mathbf{U}}_n^-$ and $\widehat{\mathbf{H}}_l$ on $\bigotimes^l$ commute.*

Proof. Since the coproduct is coassociative (see [G1, Theorem 1(ii)]), we are reduced to the case $l=2$. By definition

$$(c) \qquad \Delta(\mathbf{f}_\alpha) = \sum_{\alpha=\beta+\gamma} v^{n(\gamma,\beta)} \mathbf{f}_\beta \mathbf{k}_\gamma \otimes \mathbf{f}_\gamma.$$

Thus $\mathbf{f}_\alpha$ acts on $\bigotimes^2$ as

$$\begin{cases} \mathbf{f}_{\bar\imath} \otimes 1 + \mathbf{k}_{\bar\imath} \otimes \mathbf{f}_{\bar\imath} & \text{if} \quad \alpha = \epsilon_{\bar\imath} \\ \mathbf{f}_{\bar\imath} \otimes \mathbf{f}_{\bar\jmath} + \mathbf{f}_{\bar\jmath} \otimes \mathbf{f}_{\bar\imath} & \text{if} \quad \alpha = \epsilon_{\bar\imath} + \epsilon_{\bar\jmath} \quad \text{and} \quad \bar\imath \neq \bar\jmath \\ \mathbf{f}_{\bar\imath} \otimes \mathbf{f}_{\bar\imath} & \text{if} \quad \alpha = 2\epsilon_{\bar\imath} \\ 0 & \text{else.} \end{cases}$$

The commutation results from a direct computation. □

8.3. Lemma. *The map* $\otimes \mathrm{x}_{\mathbf{i}} \mapsto v^{l(\omega_{\mathbf{i}})} e_{\mathbf{i}}$, *for all* $\mathbf{i} \in A_l^n$, *extends uniquely to an isomorphism of* $\mathbf{U}_n^- \times \widehat{\mathbf{H}}_l$*-bimodules* $\bigotimes^l \tilde{\to} \mathbf{T}_{n,l}$.

Proof. The map above extends uniquely to an isomorphism of $\widehat{\mathbf{H}}_l$-modules. Let us prove that this isomorphism commutes to $\mathbf{U}_n^-$. For any $\mathbf{i} \in \mathbb{Z}^l$ Lemma 8.1 and formula (c) give

$$\mathbf{f}_\alpha(\otimes \mathrm{x}_{\mathbf{i}}) = \textstyle\sum_{\mathbf{n}} v^{c(\mathbf{i},\mathbf{i}+\mathbf{n})} \otimes \mathrm{x}_{\mathbf{i}+\mathbf{n}},$$

where $\mathbf{n} = (n_1, n_2, ..., n_l) \in \{0,1\}^l$ describes the set of all sequences such that $\alpha = \sum_{s=1}^l n_s \epsilon_{\bar{\imath}_s}$ and

$$c(\mathbf{i}, \mathbf{i}+\mathbf{n}) = -\sum_{1 \le s < t \le l} n_t(1-n_s) n(\epsilon_{\bar{\imath}_t}, \epsilon_{\bar{\imath}_s}).$$

By Proposition 7.4, after the specialisation $v = q^{-1}$ we have $q^{-l(\omega_{\mathbf{i}})} \mathbf{f}_\alpha(e_{\mathbf{i}}) = (\Phi \circ \Theta)(\mathbf{f}_\alpha) \star \mathbf{1}_{\mathbf{i}}$. The R.H.S. is the convolution product of $\mathbf{1}_{\mathbf{i}}$ and a function supported by the set of all pairs (L', L) such that for all i we have

$$(d) \quad L'_i \subseteq L_i \subseteq L'_{i+1}, \qquad \dim_{\mathbb{F}}(L_i/L'_i) = \alpha_i, \quad \text{and} \quad \dim_{\mathbb{F}}(L_i/L_{i-1}) = \sharp \mathbf{i}^{-1}(i).$$

By definition of the convolution product $(\Phi \circ \Theta)(\mathbf{f}_\alpha) \star \mathbf{1}_{\mathbf{i}}$ (simply denoted by $\mathbf{f}_\alpha(\mathbf{1}_{\mathbf{i}})$) is a linear combination of the $\mathbf{1}_{\mathbf{j}}$'s such that it exists $L \in Y$ such that $(L, L_\emptyset) \in X_{\mathbf{i}}$ and $(L_{\mathbf{j}}, L)$ satisfies (d). Suppose first that $\mathbf{i} \in A_l^n$. Then $X_{\mathbf{i}} \cap (Y \times \{L_\emptyset\}) = \{(L_{\mathbf{i}}, L_\emptyset)\}$. Thus $\mathbf{f}_\alpha(\mathbf{1}_{\mathbf{i}})$ is a linear combination of the $\mathbf{1}_{\mathbf{j}}$'s such that

$$(e) \qquad \mathbf{i} \le \mathbf{j} \le \mathbf{i}+1 \quad \text{and} \quad \alpha_i = \sharp(\mathbf{i}^{-1}(i) \cap \mathbf{j}^{-1}(i+1)).$$

More precisely we get

$$\mathbf{f}_\alpha(\mathbf{1}_{\mathbf{i}}) = \sum_{\mathbf{n}} q^{-a(\mathbf{i}+\mathbf{n},\mathbf{i}) - l(\omega_{\mathbf{i}}) + l(\omega_{\mathbf{i}+\mathbf{n}})} \mathbf{1}_{\mathbf{i}+\mathbf{n}},$$

where the $\mathbf{n}$'s are as above and $a(\mathbf{i}+\mathbf{n}, \mathbf{i}) = \sum_i \alpha_i (\sharp \mathbf{i}^{-1}(i+1) - \alpha_{i+1})$. A simple computation using (e) gives

$$a(\mathbf{i}+\mathbf{n}, \mathbf{i}) = \sum_{s,t=1}^{l} n_t(1-n_s) \delta_{\bar{\imath}_t+1, \bar{\imath}_s}.$$

Moreover for any sequence $\mathbf{j}$ we have $l(\omega_{\mathbf{j}}) = \dim(P_{\mathbf{j}}/(B \cap P_{\mathbf{j}}))$ where $P_{\mathbf{j}}, B \subset G$ are the isotropy subgroup of $L_{\mathbf{j}}$ and $L_\emptyset$. Thus we obtain

$$l(\omega_{\mathbf{i}+\mathbf{n}}) - l(\omega_{\mathbf{i}}) = \sum_{1 \le t < s \le l} n_t(1-n_s) \delta_{\bar{\imath}_s, \bar{\imath}_t+1}(1-\delta_{i_t,0}) +$$

$$+ \sum_{1 \le t < s \le l} n_s(1-n_t) \delta_{i_t, 1-n} \delta_{i_s,0} - \sum_{1 \le t < s \le l} n_s(1-n_t) \delta_{\bar{\imath}_t, \bar{\imath}_s}.$$

To conclude it suffices to compute the image of $\otimes \mathrm{x}_{\mathbf{i}+\mathbf{n}}$ in $\mathbb{C}_G(Y\times X)$. Using the identity

$$X_k^{-1}=\tilde{T}_{k-1}^{-1}\cdots\tilde{T}_1^{-1}P\tilde{T}_{l-1}\cdots\tilde{T}_k$$

and Lemmas 7.3 and 8.2, we get that $\otimes \mathrm{x}_{\mathbf{i}+\mathbf{n}}$ is mapped to

$$q^{-d(\mathbf{i},\mathbf{i}+\mathbf{n})}\mathbf{1}_{\mathbf{i}+\mathbf{n}},\quad d(\mathbf{i},\mathbf{i}+\mathbf{n})=\sharp\{1\leq s,t\leq l\,|\,i_s=0,\,i_t=1-n,\,n_s=1,\,n_t=0\}.$$

Then, the equality results from an easy computation. The general case (i.e. $\mathbf{i}\notin A_l^n$) follows since the isomorphism we consider commutes to $\widehat{\mathbf{H}}_l$. □

8.4. Let ψ be the semilinear involution on $\mathbf{T}_{n,l}\simeq\bigotimes^l$ such that $\psi(e_\mathbf{i}t)=\bar{e}_\mathbf{i}\bar{t}$ for all $t\in\widehat{\mathbf{H}}_l$. Proposition 7.6 and the definition of the involutions ϕ and ψ imply the following lemma (see Subsection 7.6).

Lemma. *For all $u\in\mathbf{U}_n^-$ and all $t\in\bigotimes^l$ we have $\psi(u\,t)=\bar{u}\psi(t)$.* □

9. The action of $\mathbf{U}_n^-$ on wedges.

9.1. Set $\Omega^l=\sum_i\mathrm{Im}\,(T_i+1)\subset\bigotimes^l$. We have

$$\bigotimes{}^l/\Omega^l\simeq\bigoplus_{\mathbf{i}\in A_l^n}e_\mathbf{i}\,\widehat{\mathbf{H}}_l\,e^-,$$

where $e^-=\sum_{x\in\mathfrak{S}_l}(-v)^{l(x)}\tilde{T}_x$. For any $\mathbf{i}\in\mathbb{Z}^l$ let $\wedge\mathrm{x}_\mathbf{i}$ be the image of $\otimes\mathrm{x}_\mathbf{i}$ in $\bigotimes^l/\Omega^l$. Set

$$P_l^{++}=\{\mathbf{i}\in\mathbb{Z}^l\,|\,i_1>i_2>...>i_l\}.$$

The $\wedge\mathrm{x}_\mathbf{i}$'s such that $\mathbf{i}\in P_l^{++}$ form a basis of $\bigotimes^l/\Omega^l$ (see [KMS, Proposition 1.3]). For any $\lambda\in\Pi_l$ set $|\lambda\rangle=\wedge\mathrm{x}_\mathbf{i}$ if $\mathbf{i}=\lambda+\rho$, where ρ is as in Section 1.1. Let $\bigwedge^l\subset\bigotimes^l/\Omega^l$ be the linear span of the vectors $|\lambda\rangle$.

9.2. The representation of $\mathbf{U}_n^-$ on $\bigotimes^l$ descends to $\bigotimes^l/\Omega^l$ (use Lemma 8.2). For all $\lambda\in\Pi_l$ set $\mathbf{b}_\lambda=\mathbf{b}_{O_\lambda}|\emptyset\rangle$ and put $\mathbf{B}_l=\{\mathbf{b}_\lambda\,|\,\lambda\in\Pi_l\}$. Let consider the involution ψ on $\bigotimes^l/\Omega^l$ such that

$$\psi(e_\mathbf{i}\,h\,e^-)=v^{2l(\omega)}\bar{e}_\mathbf{i}\,\bar{h}\,e^-\qquad\forall h\in\widehat{\mathbf{H}}_l.$$

Proposition. $\mathbf{B}_l$ *is a basis of* $\bigwedge^l$ *whose elements are fixed by* ψ.

Proof. Lemma 8.1, the definition of Δ, and the normal ordering rule [KMS, (43) and (45)] imply that for any $\lambda\in\Pi_l$ and any orbit $O\subset\bar{O}_\lambda\setminus O_\lambda$ we have

$$\mathbf{f}_{O_\lambda}(|\emptyset\rangle)\in v^{\mathbb{Z}}|\lambda\rangle+\bigoplus\nolimits_{\mu<\lambda}\mathbb{A}|\mu\rangle\quad\text{and}\quad\mathbf{f}_O(|\emptyset\rangle)\in\bigoplus\nolimits_{\mu<\lambda}\mathbb{A}|\mu\rangle.$$

Thus, $\mathbf{B}_l$ is a basis. Now Lemma 8.4 implies that the action of $\mathbf{b}_{O_\mathbf{m}}$ on $\bigwedge^l$ commutes to ψ. Since $\psi(|\emptyset\rangle)=|\emptyset\rangle$ we get $\psi(\mathbf{b}_\lambda)=\mathbf{b}_\lambda$ for all λ. □

9.3. Let $P_l^+\subset\mathbb{Z}^l$ be the subset of integral dominant weights. Let $\mathfrak{S}^{\mathbf{i},l}$ be the set of minimal length representatives of the cosets in $\mathfrak{S}_\mathbf{i}\setminus\widehat{\mathfrak{S}}_l/\mathfrak{S}_l$. Thus $\mathfrak{S}^{\mathbf{i},l}=\mathfrak{S}^\mathbf{i}\cap\mathfrak{S}^l$

where $\mathfrak{S}^l$ is the set of minimal length representatives of the cosets in $\widehat{\mathfrak{S}}_l/\mathfrak{S}_l$. For any $x \in \widehat{\mathfrak{S}}_l$ let $\check{x}$ be the smallest element in the double coset $\mathfrak{S}_{\mathbf{i}} x \mathfrak{S}_l$. Set

$$\mathfrak{S}(\mathbf{i}, l) = \{x \in \mathfrak{S}^{\mathbf{i},l} \,|\, S_{\mathbf{i}} x \cap x S_l = \emptyset\}.$$

Lemma. *Fix* $\mathbf{i} \in A_l^n$. *Then,*
(a) $\quad e_{\mathbf{i}} \tilde{T}_x e^- \neq 0 \Rightarrow x \in \mathfrak{S}_{\mathbf{i}} \mathfrak{S}(\mathbf{i}, l) \mathfrak{S}_l,$
(b) $\quad (x \in \mathfrak{S}^{\mathbf{i}} \quad \& \quad (\mathbf{i})x \in P_l^{++}) \Rightarrow e_{\mathbf{i}} \tilde{T}_x e^- = v^{-l(\omega_{\mathbf{i}})} \wedge \mathrm{x}_{(\mathbf{i})x},$
(c) $\quad (\mathbf{i})x \in P_l^{++} \iff x \in \mathfrak{S}_{\mathbf{i}} \mathfrak{S}(\mathbf{i}, l) \omega.$

Proof. Suppose that $x \in \mathfrak{S}^{\mathbf{i},l}$, $s_i \in S_{\mathbf{i}}$, and $s \in S_l$, are such that $s_i x = xs$. Then

$$v^{-1} e_{\mathbf{i}} \tilde{T}_x e^- = e_{\mathbf{i}} \tilde{T}_{s_i} \tilde{T}_x e^- = e_{\mathbf{i}} \tilde{T}_x \tilde{T}_s e^- = -v e_{\mathbf{i}} \tilde{T}_x e^-.$$

Thus $e_{\mathbf{i}} \tilde{T}_x e^- = 0$. Any $x \in \widehat{\mathfrak{S}}_l$ decomposes in $x = x_{\mathbf{i}} \check{x} x_l$ where $x_{\mathbf{i}} \in \mathfrak{S}_{\mathbf{i}}$, $\check{x} \in \mathfrak{S}^{\mathbf{i},l}$, $x_l \in \mathfrak{S}_l$, and $l(x) = l(x_{\mathbf{i}}) + l(\check{x}) + l(x_l)$. In particular

$$e_{\mathbf{i}} \tilde{T}_x e^- = v^{-l(x_{\mathbf{i}})} (-v)^{l(x_l)} e_{\mathbf{i}} \tilde{T}_{\check{x}} e^-.$$

Claim (a) follows. Let us prove claim (b). Recall that if λ is dominant then $\tilde{T}_\lambda^{-1}$ is mapped to $X^\lambda = X_1^{\lambda_1} X_2^{\lambda_2} \cdots X_l^{\lambda_l}$ by the Bernstein isomorphism. Then $(8.2.b)$ implies that

$$(\otimes \mathrm{x}_{\mathbf{i}}) \tilde{T}_\lambda = \otimes \mathrm{x}_{(\mathbf{i})\lambda} \qquad \forall \lambda \in P_l^+ \quad \forall \mathbf{i} \in \mathbb{Z}^l,$$

Moreover $(8.2.a)$ implies that

$$(\otimes \mathrm{x}_{\mathbf{i}}) \tilde{T}_x = \otimes \mathrm{x}_{(\mathbf{i})x} \qquad \forall x \in \mathfrak{S}_l \cap \mathfrak{S}^{\mathbf{i}} \quad \forall \mathbf{i} \in A_l^n.$$

Fix $x \in \widehat{\mathfrak{S}}_l$. Then x decomposes uniquely as $x = y\lambda$ where $y \in \mathfrak{S}_l$ and $\lambda \in \mathbb{Z}^l$. If $(\mathbf{i})x = (\mathbf{i})y + n\lambda \in P_l^{++}$ then $\lambda \in P_l^+$. Since $s\lambda > \lambda$ for all $s \in S_l$ if λ is dominant, we get $\tilde{T}_x = \tilde{T}_y \tilde{T}_\lambda$. Suppose moreover that $x \in \mathfrak{S}^{\mathbf{i}}$. Then for any $s \in S_{\mathbf{i}}$ we have $sy\lambda > y\lambda$. Since $l(z\lambda) = l(z) + l(\lambda)$ for any $z \in \mathfrak{S}_l$ (λ is dominant), we obtain that $y \in \mathfrak{S}_l \cap \mathfrak{S}^{\mathbf{i}}$. Hence, Section 8.2 implies that

$$e_{\mathbf{i}} \tilde{T}_x e^- = e_{\mathbf{i}} \tilde{T}_y \tilde{T}_\lambda e^- = v^{-l(\omega_{\mathbf{i}})} \wedge \mathrm{x}_{(\mathbf{i})x}.$$

Finally, claim (c) follows from

$$(x \in \mathfrak{S}^{\mathbf{i}} \quad \& \quad (\mathbf{i})x \in P_l^{++}) \iff (s_i x > x > xs \quad \forall s \in S_l \quad \forall s_i \in S_{\mathbf{i}})$$
$$\iff x \in \mathfrak{S}(\mathbf{i}, l)\omega.$$

□

Proposition. *If* $\mathbf{i} \in A_l^n$ *and* $(\mathbf{i})x \in P_l^{++}$ *then* $\psi(\wedge \mathrm{x}_{(\mathbf{i})x}) = (-1)^{l(\omega)} v^{l(\omega^{\mathbf{i}})} \wedge \mathrm{x}_{(\mathbf{i})x\omega}$.

Proof. First recall that if $\lambda \in P_l^+$ and $\lambda^* = -\omega(\lambda)$, then $\tilde{T}_{\lambda^*} = \tilde{T}_\omega^{-1} \tilde{T}_{-\lambda} \tilde{T}_\omega$ (indeed, since λ, λ^* are dominant weights we have $\tilde{T}_\omega \tilde{T}_{\lambda^*} = \tilde{T}_{-\lambda\omega} = \tilde{T}_{-\lambda} \tilde{T}_\omega$. In particular,

$\tilde{T}_{-\lambda} = \tilde{T}_\omega \tilde{T}_{\lambda^*} \tilde{T}_\omega^{-1}$). Fix $\mathbf{i} \in A_l^n$ and $x \in \mathfrak{S}^{\mathbf{i}}$ such that $(\mathbf{i})x \in P_l^{++}$. As above fix $x = y\lambda$ with $y \in \mathfrak{S}_l \cap \mathfrak{S}^{\mathbf{i}}$ and $\lambda \in P_l^+$. Using Lemma 9.3.*b* we get

$$\psi(\wedge \mathrm{x}_{(\mathbf{i})x}) = \psi((\otimes \mathrm{x}_{\mathbf{i}})\tilde{T}_y \tilde{T}_\lambda e^-) = v^{-l(\omega_{\mathbf{i}})} \psi(e_{\mathbf{i}} \tilde{T}_y \tilde{T}_\lambda e^-) = v^{l(\omega_{\mathbf{i}})+2l(\omega)} e_{\mathbf{i}} \tilde{T}_{y^{-1}}^{-1} \tilde{T}_{-\lambda}^{-1} e^- =$$

$$= v^{l(\omega_{\mathbf{i}})+2l(\omega)} e_{\mathbf{i}} \tilde{T}_{y^{-1}}^{-1} \tilde{T}_\omega \tilde{T}_{\lambda^*}^{-1} \tilde{T}_\omega^{-1} e^- = (-1)^{l(\omega)} v^{2l(\omega)-l(\omega^{\mathbf{i}})} e_{\mathbf{i}} \tilde{T}_{y\omega} \tilde{T}_{\lambda^*}^{-1} e^-.$$

For all $s \in S_{\mathbf{i}}$, $sy > y$ implies that $sy\omega < y\omega$. Thus

$$\psi(\wedge \mathrm{x}_{(\mathbf{i})x}) = (-1)^{l(\omega)} v^{l(\omega^{\mathbf{i}})} (\otimes \mathrm{x}_{(\mathbf{i})y\omega}) \tilde{T}_{\lambda^*}^{-1} e^- = (-1)^{l(\omega)} v^{l(\omega^{\mathbf{i}})} \wedge \mathrm{x}_{(\mathbf{i})x\omega}.$$

□

9.4. Fix $x \in \mathfrak{S}^l$. The element

$$D_{x\omega} = \sum_{\substack{y \in \mathfrak{S}^l \\ y \leq x}} (-v)^{l(y)-l(x)} \bar{P}_{y\omega, x\omega} \tilde{T}_{y\omega} e^-$$

is fixed by the involution on $\widehat{\mathbf{H}}_l e^-$ such that $he^- \mapsto v^{2l(\omega)} \bar{h} e^-$ and

$$D_{x\omega} - \tilde{T}_{x\omega} e^- \in \bigoplus\nolimits_{y \in \mathfrak{S}_l} v^{-1} \mathbb{S}\, \tilde{T}_{y\omega} e^-$$

(see [D1]). If $\mathbf{i} \in A_l^n$ and $x \in \mathfrak{S}(\mathbf{i}, l)$ set $\mathbf{b}^-_{(\mathbf{i})x\omega} = v^{l(\omega_{\mathbf{i}})} e_{\mathbf{i}} D_{x\omega}$. Lemma 9.3 gives

$$\mathbf{b}^-_{(\mathbf{i})x\omega} = \sum_{\substack{y \in \mathfrak{S}^l \\ y \leq x}} (-v)^{l(y)-l(x)} v^{-l(y_{\mathbf{i}})} \bar{P}_{y\omega, x\omega} \wedge \mathrm{x}_{(\mathbf{i})y\omega}.$$

Similarly fix $\mathbf{i} \in A_l^n$ and $x \in \mathfrak{S}^{\mathbf{i}}$. Then

$$C'_{\omega_{\mathbf{i}} x} = v^{l(x)+l(\omega_{\mathbf{i}})} \sum_{\substack{y \in \mathfrak{S}^{\mathbf{i}} \\ y \leq x}} \sum_{z \in \mathfrak{S}_{\mathbf{i}}} P_{zy, \omega_{\mathbf{i}} x} T_{zy}.$$

If $y \in \mathfrak{S}^{\mathbf{i}}$ and $y \leq x$ then $P_{zy, \omega_{\mathbf{i}} x} = P_{\omega_{\mathbf{i}} y, \omega_{\mathbf{i}} x}$ for all $z \in \mathfrak{S}_{\mathbf{i}}$ (see [D1, page 491]). Thus

$$C'_{\omega_{\mathbf{i}} x} = v^{l(x)+l(\omega_{\mathbf{i}})} e_{\mathbf{i}} \sum_{\substack{y \in \mathfrak{S}^{\mathbf{i}} \\ y \leq x}} P_{\omega_{\mathbf{i}} y, \omega_{\mathbf{i}} x} T_y.$$

If $x \in \mathfrak{S}(\mathbf{i}, l)$ set $\mathbf{b}^+_{(\mathbf{i})x\omega} = (-v)^{l(\omega)} C'_{\omega_{\mathbf{i}} x} e^-$. Then Lemma 2.2 gives

$$\begin{aligned} \mathbf{b}^+_{(\mathbf{i})x\omega} &= \textstyle\sum_{(y,z)} v^{l(x)-l(yz)} (-v)^{l(z)} P_{\omega_{\mathbf{i}} yz, \omega_{\mathbf{i}} x} \wedge \mathrm{x}_{(\mathbf{i})y\omega} \\ &= \textstyle\sum_{\substack{y \in \mathfrak{S}(\mathbf{i},l) \\ y \leq x}} v^{l(x)-l(y)} Q_{\omega_{\mathbf{i}} y, \omega_{\mathbf{i}} x} \wedge \mathrm{x}_{(\mathbf{i})y\omega}, \end{aligned}$$

where the first sum is over all couples $(y, z) \in \mathfrak{S}(\mathbf{i}, l) \times \mathfrak{S}_l$ such that $yz \leq x$ and $Q_{\omega_{\mathbf{i}} y, \omega_{\mathbf{i}} x} = \sum_z (-1)^{l(z)} P_{\omega_{\mathbf{i}} yz, \omega_{\mathbf{i}} x}$ is a parabolic Kazhdan-Lusztig polynomial. Observe that $\mathbf{b}^\pm_{(\mathbf{i})x\omega}$ is completely characterized by the following properties :

$$\psi(\mathbf{b}^\pm_{(\mathbf{i})x\omega}) = \mathbf{b}^\pm_{(\mathbf{i})x\omega}, \qquad \mathbf{b}^-_{(\mathbf{i})x\omega} - \wedge \mathrm{x}_{(\mathbf{i})x\omega} \in \bigoplus_{\substack{y \in \mathfrak{S}(\mathbf{i},l) \\ y < x}} v^{-1} \mathbb{S} \wedge \mathrm{x}_{(\mathbf{i})y\omega},$$

$$\text{and}\quad \mathbf{b}^{+}_{(\mathbf{i})x\omega} - \wedge \mathrm{x}_{(\mathbf{i})x\omega} \in \bigoplus_{\substack{y\in\mathfrak{S}(\mathbf{i},l)\\ y<x}} v\,\mathbb{S}\wedge \mathrm{x}_{(\mathbf{i})y\omega}.$$

In particular $\{\mathbf{b}^-_{\mathbf{i}}\,|\,\mathbf{i}\in P_l^{++}\}$ and $\{\mathbf{b}^+_{\mathbf{i}}\,|\,\mathbf{i}\in P_l^{++}\}$ are bases of $\bigotimes^l/\Omega^l$. For all $\lambda\in\Pi_l$ set $\mathbf{b}^\pm_\lambda=\mathbf{b}^\pm_{\mathbf{i}}$ if $\mathbf{i}=\lambda+\rho$. Put $\mathbf{B}^\pm_l=\{\mathbf{b}^\pm_\lambda\,|\,\lambda\in\Pi_l\}$.

Remark. If $\mathbf{i}\in A^n_l$ and $x,y\in\mathfrak{S}^{\mathbf{i}}$ are such that if $(\mathbf{i})x,(\mathbf{i})y\in P_l^{++}$ then

$$y\leq x\Rightarrow(\mathbf{i})x-(\mathbf{i})y \text{ is a positive root.}$$

9.5. The space $\bigwedge^l$ is endowed with four bases : $\mathbf{B}^\pm_l=\{\mathbf{b}^\pm_\lambda\,|\,\lambda\in\Pi_l\}$, $\mathbf{B}_l=\{\mathbf{b}_\lambda\,|\,\lambda\in\Pi_l\}$, and $\{|\lambda\rangle\,|\,\lambda\in\Pi_l\}$. Moreover, $\mathbf{B}^\pm_l$ are characterized by

$$\psi(\mathbf{b}^\pm_\lambda)=\mathbf{b}^\pm_\lambda,\qquad \mathbf{b}^-_\lambda-|\lambda\rangle\in\bigoplus_{\mu<\lambda}v^{-1}\mathbb{S}|\mu\rangle\quad\text{and}\quad \mathbf{b}^+_\lambda-|\lambda\rangle\in\bigoplus_{\mu<\lambda}v\,\mathbb{S}|\mu\rangle$$

(in particular $\psi(\bigwedge^l)=\bigwedge^l$). Recall that if $x\in\widehat{\mathfrak{S}}_l$ and $\lambda\in\mathbb{Z}^l$, then $\lambda\cdot x=(\lambda+\rho)x-\rho$ (see 1.1). Section 9.4 implies the following theorem.

Theorem. (a) *If $\lambda\in\Pi_l$ and x is minimal such that $\mathbf{i}=\lambda\cdot x^{-1}+\rho\in A^n_l$, then*

$$\mathbf{b}^-_\lambda=\sum_y(-v)^{l(y)-l(x)}v^{-l(y_{\mathbf{i}})}\bar{P}_{yx}|\lambda\cdot x^{-1}y\rangle,$$

where the sum is over all the y such that $y\leq x$ and $\lambda\cdot x^{-1}y\in\Pi_l$.
(b) *For all $\lambda\in\Pi_l$ the coordinates of $\mathbf{b}^+_\lambda$ in the wedges are some parabolic Kazhdan-Lusztig polynomials (w.r.t. the parabolic subgroup $\mathfrak{S}_l\subset\widehat{\mathfrak{S}}_l$).* □

Now, suppose that $l\leq n$. Let consider $\mathbf{i},\mathbf{j}\in A^n_l$, $\mathbf{n}\in M^+$ and $\mathbf{m}=\mathbf{n}^{\mathbf{j}}\in M_{\mathbf{ji}}\cap M^+$. If $x\in\mathfrak{S}^{\mathbf{i}}$ is such that $(\mathbf{i})x\in P_l^{++}$ then Section 7 gives

$$\mathbf{f}_{O_{\mathbf{n}}}(\wedge\mathrm{x}_{(\mathbf{i})x})=v^{l(\omega_{\mathbf{i}})+y(\mathbf{m})}T_{\mathbf{m}}\tilde{T}_xe^-\in e_{\mathbf{j}}\widehat{\mathbf{H}}_le^-.$$

In particular if $\mathbf{i}=(\rho)\omega\in A^n_l$ and $x=\omega$ we get

$$\mathbf{f}_{O_{\mathbf{n}}}(\wedge\mathrm{x}_\emptyset)=v^{l(\omega_{\mathbf{j}})}e_{\mathbf{j}}\tilde{T}_m\tilde{T}_\omega e^-,$$

where $m\in\mathbf{m}$ is the smallest element. Let suppose that $\mathbf{f}_{O_{\mathbf{n}}}(\wedge\mathrm{x}_\emptyset)\neq0$. Let $\mathfrak{S}^l$ be the set of minimal length representatives of the cosets in $\widehat{\mathfrak{S}}_l/\mathfrak{S}_l$. Then Lemma 9.3 implies that $m=yt$ with $t\in\mathfrak{S}_l$ and $y\in\mathfrak{S}^{\mathbf{j}}\cap\mathfrak{S}^l$ such that $S_{\mathbf{j}}y\cap yS_l=\emptyset$. Then,

$$\mathbf{f}_{O_{\mathbf{n}}}(\wedge\mathrm{x}_\emptyset)=v^{l(\omega_{\mathbf{j}})}e_{\mathbf{j}}\tilde{T}_y\tilde{T}_t\tilde{T}_\omega e^-=v^{l(\omega_{\mathbf{j}})}e_{\mathbf{j}}\tilde{T}_y\tilde{T}_\omega\tilde{T}_{\omega t\omega}e^-=(-v)^{l(t)}\wedge\mathrm{x}_{(\mathbf{j})y\omega}\in\bigoplus_\lambda\mathbb{S}\,|\lambda\rangle.$$

As a consequence if $l\leq n$ then $\mathbf{B}^+_l=\mathbf{B}_l$.

Conjecture. *The bases $\mathbf{B}_l$ and $\mathbf{B}^+_l$ coincide for all l.* □

10. Proof of Theorem 6.3.

10.1. Let $\bigotimes^\infty$ be the free $\mathbb{A}$-module linearly generated by the semi-infinite monomials

$$\otimes \mathrm{x}_{\mathbf{i}} = \mathrm{x}_{i_1} \otimes \mathrm{x}_{i_2} \otimes \mathrm{x}_{i_3} \otimes \cdots$$

where $\mathbf{i} = (i_1, i_2, ...)$ is a sequence of integers such that $i_k = 1 - k$ for $k >> 1$. The affine Hecke algebra of type $\mathfrak{gl}_\infty$ acts on $\bigotimes^\infty$ via formulas (8.b) and (8.c). Set $\Omega^\infty = \sum_i \mathrm{Im}\,(T_i + 1) \subset \bigotimes^\infty$. As above $\wedge \mathrm{x}_{\mathbf{i}}$ is the class of $\otimes \mathrm{x}_{\mathbf{i}}$ in $\bigotimes^\infty / \Omega^\infty$. The formulas in Section 8 and [KMS, Lemma 2.2] imply that for all $\alpha \in \mathbb{N}^{\mathbb{Z}/n\mathbb{Z}}$ we have

$$(a) \qquad \forall \mathbf{i} \quad \exists l \in \mathbb{N}^\times \quad \text{such that} \quad \mathbf{f}_\alpha(\wedge \mathrm{x}_{\mathbf{i}}) = \mathbf{f}_\alpha(\mathrm{x}_{i_1} \wedge \cdots \wedge \mathrm{x}_{i_l}) \wedge \mathrm{x}_{i_{l+1}} \wedge \mathrm{x}_{i_{l+2}} \wedge ...$$

Thus the action of $\mathbf{U}_n^-$ on $\bigwedge^l$ induces an action on $\bigwedge^\infty$.

Lemma. *The map* $|\lambda\rangle \mapsto \wedge \mathrm{x}_{\mathbf{i}}$*, where* $i_k = 1 + \lambda_k - k$*, gives an embedding of the representation of* $\mathbf{U}_n^-$ *on* $\bigwedge^\infty$ *given in Section 6 into* $\bigotimes^\infty / \Omega^\infty$*.*

Proof. The proof goes by a direct computation. First observe that $\wedge \mathrm{x}_{\mathbf{i}}$ and $|\lambda\rangle$ have the same weight for any $\lambda \in \Pi$ if $\mathbf{i}$ is the sequence such that $i_k = 1 + \lambda_k - k$. Fix $\lambda \in \Pi$ and $\alpha \in \mathbb{N}^{\mathbb{Z}/n\mathbb{Z}}$. Fix $\mathbf{i}$ as above. Formula 6.2.a gives

$$\mathbf{f}_\alpha(\wedge \mathrm{x}_{\mathbf{i}}) = \mathbf{f}_\alpha(|\lambda\rangle) = \sum_{a \; s.t. \; \bar{a} = \alpha} \gamma_a(\mathbf{f}_\alpha) \prod_{\substack{j<i \\ \bar{\imath}=\bar{\jmath}}} \mathbf{k}_i^{a_j}(|\lambda\rangle).$$

Moreover Remark 6.1 implies that $\gamma_a(\mathbf{f}_\alpha)$ is $v^{h(a)}$ times the product of the $\mathbf{f}_i^{(a_i)}$'s ordered from $i = -\infty$ to ∞. Using the formulas in Section 4 we first observe that $\mathbf{f}_i^{(2)}$ acts by zero on the Fock space for any i. The elements $|\lambda\rangle$ and $\otimes \mathrm{x}_{\mathbf{i}}$ have the same weight with respect to $\mathbf{k}_i$. Thus we get

$$\mathbf{f}_\alpha(\wedge \mathrm{x}_{\mathbf{i}}) = \sum_{\mathbf{n}} v^{e(\mathbf{i}, \mathbf{i}+\mathbf{n})} \wedge \mathrm{x}_{\mathbf{i}+\mathbf{n}},$$

where $\mathbf{n} = (n_1, n_2, ...) \in \{0,1\}^{\mathbb{N}^\times}$ describes the set of all sequences such that $\alpha = \sum_{s \geq 1} n_s \epsilon_{\bar{\imath}_s}$ and

$$e(\mathbf{i}, \mathbf{i}+\mathbf{n}) = \sum_{i_k > i_l} n_l \delta_{\bar{\imath}_l, \bar{\imath}_k} - \sum_{i_k > 1 + i_l} n_l \delta_{\bar{\imath}_l + 1, \bar{\imath}_k} -$$

$$- \sum_{i_k < i_l} n_l n_k \delta_{\bar{\imath}_l, \bar{\imath}_k} + \sum_{1 + i_k < i_l} n_l n_k \delta_{\bar{\imath}_l, \bar{\imath}_k + 1}.$$

If $\wedge \mathrm{x}_{\mathbf{i}+\mathbf{n}} \neq 0$ then $e(\mathbf{i}, \mathbf{i}+\mathbf{n}) = \sum_{i_k > i_l} n_l (1 - n_k)(\delta_{\bar{\imath}_l, \bar{\imath}_k} - \delta_{\bar{\imath}_l + 1, \bar{\imath}_k})$. On the other hand the formula in Section 8.3 gives

$$\mathbf{f}_\alpha(\wedge \mathrm{x}_{\mathbf{i}}) = \sum_{\mathbf{n}} v^{c(\mathbf{i}, \mathbf{i}+\mathbf{n})} \wedge \mathrm{x}_{\mathbf{i}+\mathbf{n}},$$

where $\mathbf{n}$ describes the same set and

$$c(\mathbf{i}, \mathbf{i}+\mathbf{n}) = - \sum_{1 \leq k < l} n_l (1 - n_k) n(\epsilon_{\bar{\imath}_l}, \epsilon_{\bar{\imath}_k}).$$

We are through (recall that $\mathbf{i}$ is decreasing). □

Theorem 6.3 follows from Proposition 9.2 and Lemma 10.1.

10.2. The involution ψ on $\bigwedge^l$ induces the semilinear involution ψ on $\bigwedge^\infty$ such that,

$$\forall \mathbf{i}, \quad l \geq \sum_k (i_k - 1 + k) \Rightarrow \psi(\wedge \mathrm{x}_{\mathbf{i}}) = \psi(\mathrm{x}_{i_1} \wedge \cdots \wedge \mathrm{x}_{i_l}) \wedge \mathrm{x}_{i_{l+1}} \wedge \mathrm{x}_{i_{l+2}} \wedge ...$$

Proposition 9.3 implies that ψ coincides with the involution on $\bigwedge^\infty$ used in [LT]. In [LT] Leclerc and Thibon have defined two bases $\mathbf{B}^\pm = \{\mathbf{b}^\pm_\lambda \,|\, \lambda \in \Pi\}$ in $\bigwedge^\infty$ such that for all λ

$$\psi(\mathbf{b}^\pm_\lambda) = \mathbf{b}^\pm_\lambda, \qquad \mathbf{b}^-_\lambda - |\lambda\rangle \in \bigoplus_{\mu<\lambda} v^{-1}\bar{\mathbb{S}}|\mu\rangle \quad \text{and} \quad \mathbf{b}^+_\lambda - |\lambda\rangle \in \bigoplus_{\mu<\lambda} v\,\mathbb{S}|\mu\rangle.$$

Thus, Conjecture 9.5 is equivalent to

Conjecture. *The bases* $\mathbf{B}$ *and* $\mathbf{B}^+$ *coincide.* □

Set $\mathbf{b}_\lambda = \sum_\mu d_{\mu\lambda}|\mu\rangle$ and $\mathbf{b}^\pm_\lambda = \sum_\mu e^\pm_{\mu\lambda}|\mu\rangle$. Conjecture 10.2 is precisely $d_{\lambda\mu} = e^+_{\lambda\mu}$.

11. Proof of the Decomposition Conjecture.

Let ε be a n-th root of unity. The quantized Schur algebra $\mathbf{S}_{n,l}$ is the subalgebra of $\widehat{\mathbf{S}}_{n,l}$ spanned by the elements $T_{\mathbf{m}}$ with $\mathbf{m} \in \mathfrak{S}_{\mathbf{i}} \setminus \mathfrak{S}_l / \mathfrak{S}_{\mathbf{j}}$ (see Subsection 7.4). Fix $l \leq k$. Let consider the subalgebra

$$\mathbf{S}_l = \mathbf{S}_{k,l} \cap \bigoplus_{\lambda,\mu \in \Pi(l)} \widehat{\mathbf{H}}_{\mathbf{i}_\lambda \mathbf{i}_\mu}$$

where $\mathbf{i}_\lambda = (1-k)^{\lambda_1}(2-k)^{\lambda_2}...0^{\lambda_k}$ for any $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \in \Pi(l)$. We want to compute the decomposition matrices of the simple $\mathbf{S}_l$-modules under the specialization $v \mapsto \varepsilon$. The algebra $\mathbf{S}_{k,l}$ is Morita equivalent to $\mathbf{S}_l$. For any $t \in \mathbb{C}^\times$ let $\mathbf{S}_{l|t}$ be the specialization of $\mathbf{S}_l$ at $v = t$. The simple modules of $\mathbf{S}_{l|t}$ are parametrized by $\Pi(l)$. For any k let $\mathbf{U}(\mathfrak{gl}_k)$ be the Lusztig integral form of the quantized enveloping algebra of $\mathfrak{gl}_k$ and let $\mathbf{U}_\varepsilon(\mathfrak{gl}_k)$ be the specialization at $v = \varepsilon$. The set Π_k is identified with the set of dominant weights of $\mathfrak{gl}_k$ with non-negative components. If $\lambda \in \Pi_k$, let V_λ and W_λ be respectively the simple and the Weyl $\mathbf{U}_\varepsilon(\mathfrak{gl}_k)$-module with highest weight λ. There exists a surjective map $\pi : \mathbf{U}_\varepsilon(\mathfrak{gl}_k) \to \mathbf{S}_{k,l|\varepsilon}$ (see [D2]). If $\lambda \in \Pi(l)$ let L_λ, M_λ, be the simple and the Specht $\mathbf{S}_{k,l|\varepsilon}$-modules such that

$$\pi^*[L_\lambda] = [V_{\lambda'}] \quad \text{and} \quad \pi^*[M_\lambda] = [W_{\lambda'}]$$

in the Grothendieck ring.

Theorem. *The specialization at* $v = 1$ *of the matrix* $(e^+_{\lambda\mu})_{\lambda\mu}$, $\lambda, \mu \in \Pi(l)$, *is the decomposition matrix of the Specht modules of* $\mathbf{S}_l$.

Proof. The Lusztig conjecture (proved by Kashiwara-Tanisaki and Kazhdan-Lusztig) gives the multiplicity of W_μ in V_λ. More precisely we have

$$[V_\lambda : W_\mu] = \sum_y (-1)^{l(yx)} P_{yx}(1),$$

where $x \in \widehat{\mathfrak{S}}_l$ is minimal such that $\nu = \lambda \cdot x^{-1}$ satisfies

$$\nu_i < \nu_{i+1} \quad \forall i = 1, 2, ..., k-1, \qquad \nu_1 - \nu_k \geq 1 - k - n,$$

and $\mu = \lambda \cdot x^{-1} y$. According to Theorem 9.5.a, the Lusztig Conjecture is equivalent to

$$(a) \qquad [L_{\lambda'}] = \sum_{\mu} e^{-}_{\lambda\mu}(1)\, [M_{\mu'}], \qquad \forall \lambda \in \Pi_k.$$

Recall that $(e^{+}_{\lambda\mu})_{\lambda\mu} = (\bar{e}^{-}_{\lambda'\mu'})^{-1}_{\lambda\mu}$ (see [LT, Section 4]). Thus

$$(a) \iff [M_\lambda] = \sum_{\mu} e^{+}_{\lambda\mu}(1)\, [L_\mu].$$

□

12. The Lusztig conjecture.

12.1. Let F be the variety of partial flags in $\mathbb{C}^l$ of the type

$$\{0\} \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k = \mathbb{C}^l.$$

The linear group GL_l acts diagonaly on $F \times F$. Let $Z \subset T^*F \times T^*F$ be the Steinberg variety (Z is a reducible variety whose irreducible components are the closure of the conormal bundles to the GL_l-orbits in $F \times F$). The group $G = GL_l \times \mathbb{C}^\times$ acts naturally on Z : the linear group acts diagonaly and $z \in \mathbb{C}^\times$ acts by multiplication by z^{-2} along the fibers. The complexified Grothendieck group of equivariant coherent sheaves on Z, denoted by $\mathbf{K}_{k,l}$, is endowed with an associative convolution product (see [GV], [V2]) denoted by $\star$. For any $z \in \mathbb{C}^\times$, a parametrization of the simple modules of the specialized algebra $\mathbf{K}_{k,l|v=z}$ is given in [GV] (see in [V2] the remark after Theorem 4 for the case of roots of unity): the simple modules are labelled by orbits of pairs $(s, x) \in \mathrm{GL}_l \times \mathfrak{gl}_l$ where s is semi-simple, $x^k = 0$, and $sxs^{-1} = z^{-2}x$. As usual the GL_l-orbit of x is labelled by the partition $\lambda \in \Pi(l)$ such that λ_i is the length of the i-th Jordan block of x. Then $\lambda' \in \Pi(l) \cap \Pi_k$. The orbits of the pairs (s, x) such that the spectrum of s is in $z^{2\mathbb{Z}}$ are labelled by isomorphism class of nilpotent representations of Γ_∞ if z is generic and of Γ_n if $z = \varepsilon$ (recall that ε^2 is a primitive n-th root of unity). Let $\Omega_{k,l}$ and $\Omega^\infty_{k,l}$ be the corresponding sets of isomorphism classes of representations of Γ_n and Γ_∞. If $O \in \Omega^\infty_{k,l}$ (resp. $O \in \Omega_{k,l}$) let L^∞_O (resp. L_O) be the simple $\mathbf{K}_{k,l}$-module labelled by O. Similarly let M^∞_O and M_O be the standard modules labelled by O (see [V2]). Let $[M]$ be the class of the module M in the complexified Grothendieck ring. Let $\widehat{\mathbf{R}}_n$ and $\widehat{\mathbf{R}}_\infty$ be the linear span of the elements $[L_O]$ and $[L^\infty_O]$ where $O \in \Omega_{k,k}$ or $O \in \Omega^\infty_{k,k}$ and $k \geq 1$. The restricted dual $\widehat{\mathbf{R}}^*_n$ (resp. $\widehat{\mathbf{R}}^*_\infty$) is spanned by the linear forms l_O (resp. l^∞_O) such that

$$l_O([L_{O'}]) = \delta_{O,O'} \quad \text{and} \quad l^\infty_O([L^\infty_{O'}]) = \delta_{O,O'}.$$

12.2. The quantized enveloping algebra of $\widehat{\mathfrak{gl}}_k$ is generated by elements $\mathbf{e}_{i,s}$, $\mathbf{f}_{i,s}$, $\mathbf{h}_{j,t}$ and $\mathbf{k}^{\pm 1}_j$ ($0 < i < k$, $0 < j \leq k$, $s \in \mathbb{Z}$, $t \in \mathbb{Z}^\times$) which satisfy the relations of the Drinfeld new presentation. Let $\mathbf{U}(\widehat{\mathfrak{gl}}_k)$ be the $\mathbb{A}$-subalgebra generated by the elements $\mathbf{e}_{i,s}^{(m)}$, $\mathbf{f}_{i,s}^{(m)}$, $[t]^{-1}\mathbf{h}_{j,t}$ and $\mathbf{k}_j^{\pm 1}$. For any $z \in \mathbb{C}^\times$ let $\mathbf{U}_z(\widehat{\mathfrak{gl}}_k)$ be its specialization at $v = z$. In [GV], [V2], is defined a surjective algebra homomorphism $\Psi_{k,l} : \mathbf{U}(\widehat{\mathfrak{gl}}_k) \otimes_{\mathbb{A}} \mathbb{C}(v) \to \mathbf{K}_{k,l} \otimes_{\mathbb{A}} \mathbb{C}(v)$. It is proved in [S] that $\Psi_{k,l}$ restricts to a surjective homomorphism $\mathbf{U}(\widehat{\mathfrak{gl}}_k) \to \mathbf{K}_{k,l}$. Observe that the restriction of a simple $\mathbf{U}_z(\widehat{\mathfrak{gl}}_k)$-module to $\mathbf{U}_z(\widehat{\mathfrak{sl}}_k)$ is simple. Thus $\Psi_{k,l}^* L_O$ for $O \in \Omega_{k,l}$ (resp. $\Psi_{k,l}^* L_O^\infty$ for $O \in \Omega_{k,l}^\infty$), may be viewed as a simple $\mathbf{U}_z(\widehat{\mathfrak{sl}}_k)$-module when $z = \varepsilon$ (resp. z generic). Recall that there is an algebra homomorphism $ev : \mathbf{U}(\widehat{\mathfrak{sl}}_k) \to \mathbf{U}(\mathfrak{gl}_k)$ such that

$$ev(\mathbf{e}_0) = v^{-1}\{\mathbf{f}_{k-1}, \{\mathbf{f}_{k-2}, ...\{\mathbf{f}_2, \mathbf{f}_1\}...\}\}\mathbf{k}_k\mathbf{k}_{k-1}$$

$$ev(\mathbf{f}_0) = (-1)^k v^{k-1}\{\mathbf{e}_{k-1}, \{\mathbf{e}_{k-2}, ...\{\mathbf{e}_2, \mathbf{e}_1\}...\}\}\mathbf{k}_k^{-1}\mathbf{k}_{k-1}^{-1}$$

$$ev(\mathbf{f}_i) = \mathbf{f}_i, \qquad ev(\mathbf{e}_i) = \mathbf{e}_i, \qquad i = 1, 2, ..., k-1,$$

where $\{x, y\} = xy - v^{-1}yx$. If $\lambda \in \Pi_k$ let V_λ (resp. V_λ^∞) be the simple $\mathbf{U}_z(\mathfrak{gl}_k)$-modules with highest weight λ where $z = \varepsilon$ (resp. z generic). The Drinfeld polynomials of L_O and L_O^∞ are computed in [V2]. If $\lambda \in \Pi(k)$, then $\Psi_{k,k}^* L_{O_\lambda}$ and $\Psi_{k,k}^* L_{O_\lambda}^\infty$ are the pull-backs of the modules $V_{\lambda'}$ and $V_{\lambda'}^\infty$ by the evaluation map ev (see [CP, Proposition 12.2.13]). Let $\mathbf{R}_n$ and $\mathbf{R}_\infty$ be the linear span of the classes $[V_\lambda]$ and $[V_\lambda^\infty]$ for all λ and all k. The restricted dual spaces $\mathbf{R}_n^*$ and $\mathbf{R}_\infty^*$ are spanned by the linear forms l_λ and l_λ^∞ such that

$$l_\lambda([V_{\mu'}]) = \delta_{\lambda\mu} \quad \text{and} \quad l_\lambda^\infty([V_{\mu'}^\infty]) = \delta_{\lambda\mu}.$$

The element $[V_\lambda^\infty]$ may be viewed as the class in $\mathbf{R}_n$ of the Weyl module W_λ with highest weight λ. Let $s_\lambda \in \mathbf{R}_n^*$ be such that

$$s_\lambda([W_{\mu'}]) = \delta_{\lambda\mu}.$$

12.3. In this subsection $\mathbf{U}_n^-$, $\mathbf{U}_\infty^-$ and $\bigwedge^\infty$ stand for their specializations at $v = 1$.

Theorem. *The linear isomorphism* $\mathbf{R}_n^* \xrightarrow{\sim} \bigwedge^\infty$ *such that* $s_\lambda \mapsto |\lambda\rangle$ *maps* l_λ *to* $\mathbf{b}_\lambda$.

Proof. First observe that the classes of the standard modules $[M_O]$ and $[M_O^\infty]$ form a basis of the spaces $\widehat{\mathbf{R}}_n$ and $\widehat{\mathbf{R}}_\infty$. Let m_O and m_O^∞ be the elements of the dual basis. To avoid confusions let $\mathbf{f}_O^\infty$, $\mathbf{b}_O^\infty$, denote the generatrors of $\mathbf{U}_\infty^-$. The multiplicity formula [GV, Theorem 6.6] implies that there are two linear isomorphisms

$$\iota_n : \mathbf{U}_n^- \to \widehat{\mathbf{R}}_n^* \quad \text{and} \quad \iota_\infty : \mathbf{U}_\infty^- \to \widehat{\mathbf{R}}_\infty^*$$

such that

$$(a) \qquad \iota_n(\mathbf{f}_O) = m_O, \quad \iota_n(\mathbf{b}_O) = l_O, \quad \iota_\infty(\mathbf{f}_O^\infty) = m_O^\infty, \quad \iota_\infty(\mathbf{f}_O^\infty) = l_O^\infty.$$

The spaces $\mathbf{R}_n^*$ and $\mathbf{R}_\infty^*$ are identified with $\bigwedge^\infty$ via the maps

$$s_\lambda \mapsto |\lambda\rangle \quad \text{and} \quad l_\lambda^\infty \mapsto |\lambda\rangle.$$

We obtain the following commutative square

$$\begin{array}{ccccccc} \bigwedge^\infty & = & \mathbf{R}_n^* & \stackrel{\sim}{\to} & \mathbf{R}_\infty^* & = & \bigwedge^\infty \\ & & \uparrow & & \uparrow & & \\ \mathbf{U}_n^- & = & \widehat{\mathbf{R}}_n^* & \to & \widehat{\mathbf{R}}_\infty^* & = & \mathbf{U}_\infty^-, \end{array}$$

where the horizontal arrows are the dual of the specialization maps and the vertical arrows are the dual of the evaluation maps. By definition the upper arrow maps s_λ to l_λ^∞ and both elements are identified with the vacuum vector $|\lambda\rangle$. The right vertical arrow is such that

$$\mathbf{b}_O^\infty = l_O^\infty \mapsto l_\lambda^\infty = |\lambda\rangle \quad \text{if} \quad O = O_\lambda, \qquad \mathbf{b}_O^\infty \mapsto 0 \quad \text{else.}$$

By Proposition 5 it is the quotient map

$$\mathbf{U}_\infty^- \to \bigwedge^\infty, \qquad u \mapsto u(|\emptyset\rangle).$$

Suppose first that the lower horizontal arrow is the map γ introduced in Section 6. Then the left vertical arrow is the quotient map

$$\mathbf{U}_n^- \to \bigwedge^\infty, \qquad u \mapsto u(|\emptyset\rangle).$$

Hence (a) implies that the left vertical arrow maps l_{O_λ} to $\mathbf{b}_\lambda$. Since this arrow is the transpose of the evaluation map we get $l_\lambda = \mathbf{b}_\lambda$ and we are through. By Subsection 6.4, to prove that the map $\widehat{\mathbf{R}}_n^* \to \widehat{\mathbf{R}}_\infty^*$ is γ we are reduced to prove that if $r(O') \subseteq O$ then $[M_{O'}^\infty]$ specializes to $[M_O]$. This is obvious by the localization theorem in equivariant K-theory. □

12.4. Theorem 12.3 implies that $[V_\lambda^\infty] = \sum_\mu d_{\lambda'\mu'}(1)\,[V_\mu]$. According to Section 11 the Lusztig Conjecture can be written as

$$[W_\lambda] = \sum_\mu e^+_{\lambda'\mu'}(1)\,[V_\mu] \quad \forall\lambda,$$

which is precisely Conjecture 10.2.

13. Proof of Proposition 6.1.

13.1. Fix $\Gamma = \Gamma_n$ or Γ_∞. Let $\mathcal{S}_d$ be the set of finite sequences $\mathbf{d} = (d^1, d^2, ..., d^l)$ of elements in $\mathbb{N}^{(I)}$ such that $\sum_k d^k = d$. Fix a I-graded vector space V of dimension d. For each $\mathbf{d} \in \mathcal{S}_d$ let $F_\mathbf{d}$ be the set of flags of V of type $\mathbf{d}$, i.e. $F_\mathbf{d}$ is the set of filtrations $F = (\{0\} = F^0 \subseteq F^1 \subseteq \cdots \subseteq F^l = V)$ such that F^k is I-graded and has dimension $d^1 + d^2 + \cdots + d^k$. Given $x \in E_V$ we say that a flag $F \in F_\mathbf{d}$ is x-stable if $x(F^k) \subseteq F^{k-1}$ for all k. Let $\tilde{F}_\mathbf{d}$ be the variety of all pairs (x, F) such that $x \in E_V$ and $F \in F_\mathbf{d}$ is x-stable. The group G_V acts on $\tilde{F}_\mathbf{d}$ in the obvious way. Let $\pi_\mathbf{d} : \tilde{F}_\mathbf{d} \to E_V$ be the first projection. The map $\pi_\mathbf{d}$ commutes to G_V. Thus the function $f_\mathbf{d} = \pi_{\mathbf{d}\,!}(1)$ belongs to $\mathbb{C}_{G_V}(E_V)$.

Lemma. *(a) The space $\mathbb{C}_{G_V}(E_V)$ is linearly spanned by the elements $f_\mathbf{d}$ with $\mathbf{d} \in \mathcal{S}_d$.*

(b) For any $a, b \in \mathbb{N}^{(I)}$ and any $\mathbf{a} \in \mathcal{S}_a$, $\mathbf{b} \in \mathcal{S}_b$, we have $f_{\mathbf{a}} \circ f_{\mathbf{b}} = q^{-m(b,a)} f_{\mathbf{ab}}$ where $\mathbf{ab} \in \mathcal{S}_{a+b}$ is the sequence $\mathbf{a}$ followed by the sequence $\mathbf{b}$.

Proof. Claim (b) is proved as in [L2, Lemma 3.2.b]. Let us prove claim (a). If a flag F is x-stable then $F^k \subseteq \mathrm{Ker}\,(x^k)$. Thus if $\mathbf{d} \in \mathcal{S}_d$ is such that

$$d^1 + d^2 + \cdots + d^k = \dim \mathrm{Ker}\,(x^k) \quad \forall k = 1, 2, 3, ...,$$

then $\pi_{\mathbf{d}}^{-1}(x)$ is reduced to the single flag

$$\{0\} \subseteq \mathrm{Ker}\,(x) \subseteq \mathrm{Ker}\,(x^2) \subseteq \cdots \subseteq V.$$

In particular $f_{\mathbf{d}}(x) = 1$. Moreover, in this case $f_{\mathbf{d}}$ is supported on the G_V-orbits of the y's such that

$$\dim \mathrm{Ker}\,(x^k) \leq \dim \mathrm{Ker}\,(y^k) \quad \forall k = 1, 2, 3, ...,$$

i.e. $y \in \overline{G_V \cdot x}$. We are through. $\square$

Remark. It is easy to see that for any $d \in \mathbb{N}^{\mathbb{Z}/n\mathbb{Z}}$ and $\mathbf{d} = (d)$ we have $f_{\mathbf{d}}=\mathbf{f}_d$. Thus Proposition 3.5 is a consequence of (a) and (b).

13.2. We fix a $\mathbb{Z}$-graded vector space V of dimension d. Let $\bar{V}$ be the associated $\mathbb{Z}/n\mathbb{Z}$-graded vector space, of dimension $\bar{d}$. The space $\bar{V}$ is endowed with the $\mathbb{Z}$-filtration whose associated graded is identified with V. Fix $\bar{\mathbf{d}} \in \mathcal{S}_{\bar{d}}$. We have the following commutative diagram

$$\begin{array}{ccccc} \tilde{F}_{\bar{\mathbf{d}}} & \xrightarrow{\pi_{\bar{\mathbf{d}}}} & E_{\bar{V}} & & \\ \uparrow & & \uparrow j & & \\ \tilde{F}_{\bar{\mathbf{d}},d} & \xrightarrow{\pi_{\bar{\mathbf{d}},d}} & E_{\bar{V},V} & \xrightarrow{p} & E_V, \end{array}$$

where $\tilde{F}_{\bar{\mathbf{d}},d} = \pi_{\bar{\mathbf{d}}}^{-1}(E_{\bar{V},V})$ and the vertical arrows are the embeddings. We have clearly

$$(c) \qquad p_! j^*(f_{\bar{\mathbf{d}}}) = (p\pi_{\bar{\mathbf{d}},d})_!(1).$$

Let $\mathcal{S}_{\bar{\mathbf{d}},d} \subset \mathcal{S}_d$ be the set of sequences $\mathbf{d}$ such that $\sum_{i\in\bar{\imath}} d_i^k = \bar{d}_{\bar{\imath}}^k$ for any k and $\bar{\imath}$. If $\mathbf{d} \in \mathcal{S}_{\bar{\mathbf{d}},d}$ let $\tilde{F}_{\bar{\mathbf{d}},\mathbf{d}} \subset \tilde{F}_{\bar{\mathbf{d}},d}$ be the set of pairs (x, F) such that the associated graded of F^k with respect to the filtration induced by the $\mathbb{Z}$-filtration on $\bar{V}$ has dimension $d^1 + d^2 + \cdots + d^k$. The sets $\tilde{F}_{\bar{\mathbf{d}},\mathbf{d}}$ form a partition of $\tilde{F}_{\bar{\mathbf{d}},d}$. We have a commutative square

$$\begin{array}{ccc} \tilde{F}_{\bar{\mathbf{d}},d} & \xrightarrow{p\pi_{\bar{\mathbf{d}},d}} & E_V \\ \uparrow & & \uparrow \pi_{\mathbf{d}} \\ \tilde{F}_{\bar{\mathbf{d}},\mathbf{d}} & \xrightarrow{\tau} & \tilde{F}_{\mathbf{d}}, \end{array}$$

where the left vertical arrow is the inclusion and τ maps the pair (x, F) to the associated graded. Thus

$$(d) \qquad (p\pi_{\bar{\mathbf{d}},d})_!(1) = \sum_{\mathbf{d}\in\mathcal{S}_{\bar{\mathbf{d}},d}} (\pi_{\mathbf{d}}\tau)_!(1).$$

Lemma. *The map τ is a vector bundle of rank*

$$r(\mathbf{d}) = \sum_{k\geq l}\sum_{\substack{i>j\\ \bar{\imath}=\bar{\jmath}}} d_j^k d_{i+1}^l + \sum_{k<l}\sum_{\substack{i>j\\ \bar{\imath}=\bar{\jmath}}} d_j^k d_i^l.$$

Proof. The proof goes as the proof of [L2, Lemma 4.4]. More precisely fix $(x, F) \in \tilde{F}_{\mathbf{d}}$ and compute the fiber $\tau^{-1}(x, F)$. Giving a $\mathbb{Z}/n\mathbb{Z}$-graded subspace $\bar{F}^k \in \bar{V}$ of dimension $\bar{d}^1 + \bar{d}^2 + \cdots + \bar{d}^k$ whose associated $\mathbb{Z}$-graded is F^k is the same as giving a map

$$z^k = \oplus z_{ij}^k \in \bigoplus_{\substack{i>j\\ \bar{\imath}=\bar{\jmath}}} \mathrm{Hom}\,(F_j^k, V_i/F_i^k).$$

Then $\bar{F}^k \subset \bar{F}^{k+1}$ if and only if $z^{k+1} = z^k : F^k \to V/F^{k+1}$. On the other hand giving $\bar{x} \in E_{\bar{V},V}$ such that $p(\bar{x}) = x$ is the same as giving a map

$$y = \oplus y_{i+1,j} \in \bigoplus_{\substack{i>j\\ \bar{\imath}=\bar{\jmath}}} \mathrm{Hom}\,(V_j, V_{i+1}).$$

Then $\bar{F}$ is $\bar{x}$-stable if and only if

$$z_{i+1,j+1}^k \circ x_j - x_i \circ z_{ij}^k - y_{i+1,j} = 0 \,:\, F_j^k \to V_{i+1}/F_{i+1}^k.$$

The Lemma results from a direct computation. □

The Lemma and (c), (d), give

$$\gamma_d(f_{\bar{\mathbf{d}}}) = \textstyle\sum_{\mathbf{d}\in\mathcal{S}_{\bar{\mathbf{d}},d}} q^{2r(\mathbf{d})-h(d)} f_{\mathbf{d}}.$$

Fix $\alpha, \beta \in \mathbb{N}^{\mathbb{Z}/n\mathbb{Z}}$, $\bar{\mathbf{a}} \in \mathcal{S}_\alpha$, and $\bar{\mathbf{b}} \in \mathcal{S}_\beta$. Using Lemma 13.1.$b$ we get

$$\gamma_d(f_{\bar{\mathbf{a}}} \circ f_{\bar{\mathbf{b}}}) = \textstyle\sum_{\mathbf{a},\mathbf{b}} q^{m(b,a)-m(\beta,\alpha)+2r(\mathbf{ab})-h(d)} f_{\mathbf{a}} \circ f_{\mathbf{b}},$$

where the sum is over all $(\mathbf{a}, \mathbf{b}) \in \mathcal{S}_{\bar{\mathbf{a}},a} \times \mathcal{S}_{\bar{\mathbf{b}},b}$ and all (a, b) such that $\bar{a} = \alpha$, $\bar{b} = \beta$, and $d = a + b$. We are thus reduced to prove the following identity

(e) $\quad m(b, a) - m(\beta, \alpha) + 2r(\mathbf{ab}) - 2r(\mathbf{a}) - 2r(\mathbf{b}) + h(a) + h(b) - h(d) = k(b, a).$

Set

$$l_+(b, a) = \sum_{\substack{i>j\\ \bar{\imath}=\bar{\jmath}}} (b_i a_j + b_j a_{i+1}) \quad \text{and} \quad l_-(b, a) = \sum_{\substack{i<j\\ \bar{\imath}=\bar{\jmath}}} (b_i a_j + b_j a_{i+1}).$$

Then (e) follows from the following equalities which are easy to prove :

$$m(b, a) - m(\beta, \alpha) = -l_+(b, a) - l_-(b, a),$$

$$h(a) + h(b) - h(d) = k(b, a) - l_+(b, a) + l_-(b, a),$$

$$r(\mathbf{ab}) - r(\mathbf{a}) - r(\mathbf{b}) = l_+(b, a).$$

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