Number of different degree sequences of a graph with no isolated vertices

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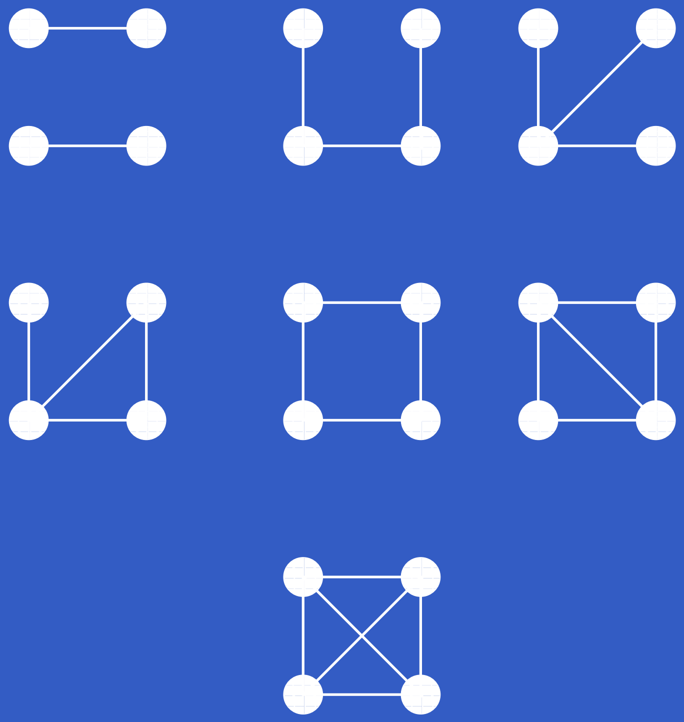
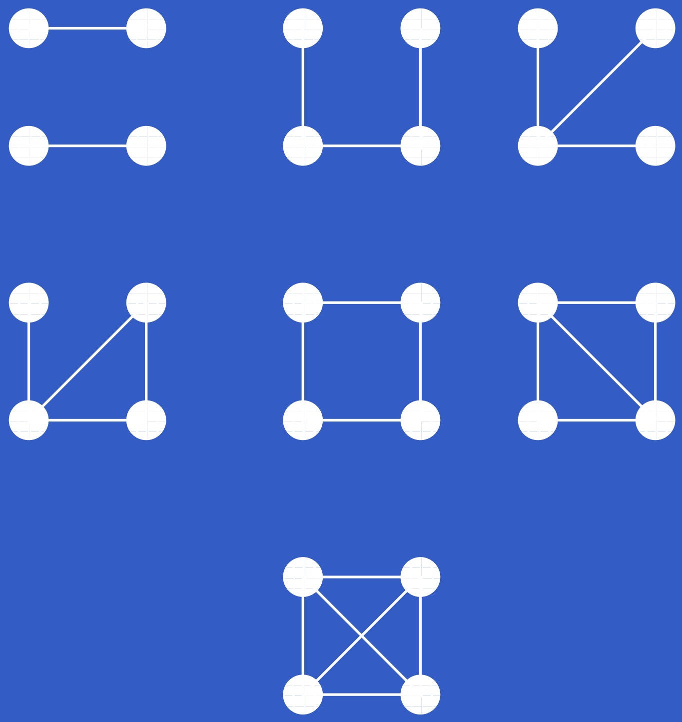
Example



Example



7 different degree seqences

Partition

A *partition* is a weakly decreasing sequence of non-negative integers, where allmost all numbers are zero.

$$\lambda = 3, 3, 2, 2, 1, 1, 0, \ldots$$

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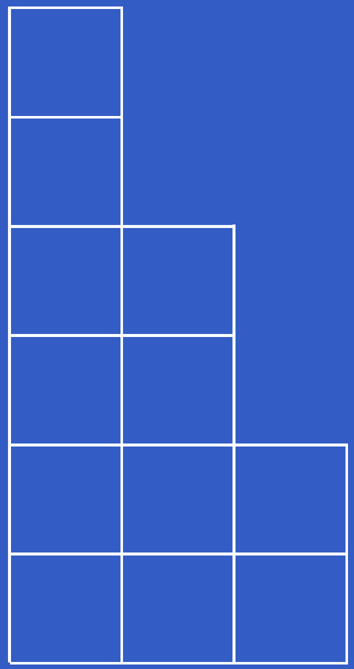
The *length* of a partition is the number of nonzero parts.

$$l(\lambda) = 6$$

Ferrers Diagram

Partitions are visualized by left adjusted boxes in the first quadrant.



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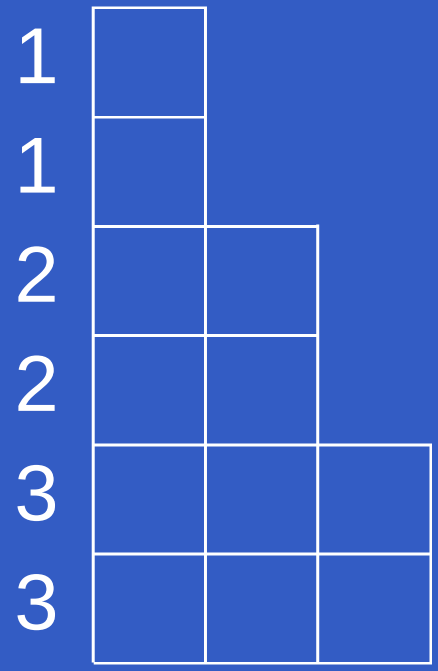
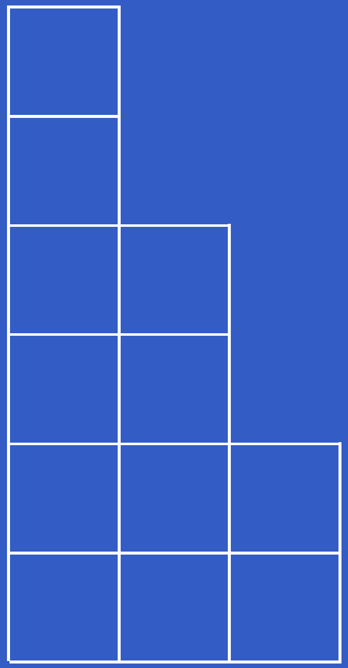


Conjugate Partition

The *conjugate* partition λ' is the sequence of numbers of boxes in the columns.



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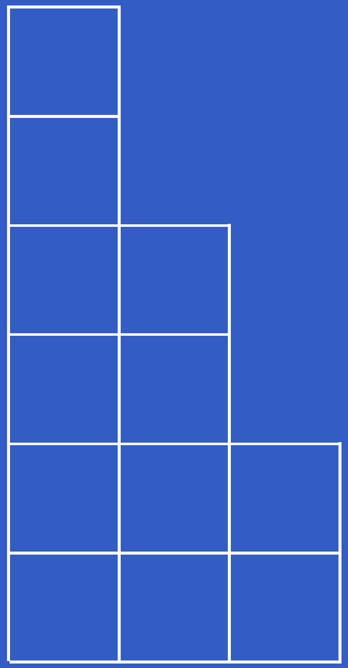

6 4 2

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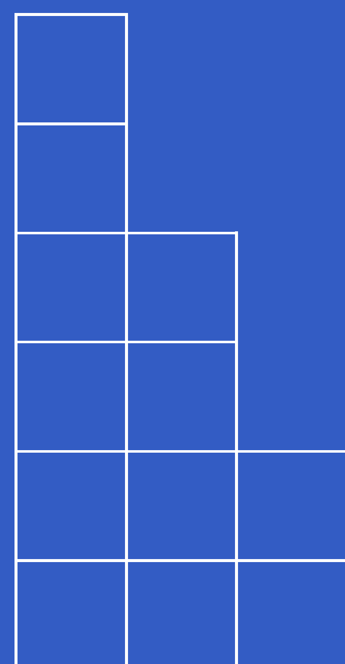

$6 \quad 4 \quad 2 \quad = (3,3,2,2,1,1)'$

Graphical Partitions

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- graphical partitions only exist for even weight
- not all even weight partitions are graphical

Example

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$$\downarrow$$

3,3,2,2,1,1

Problem

We want

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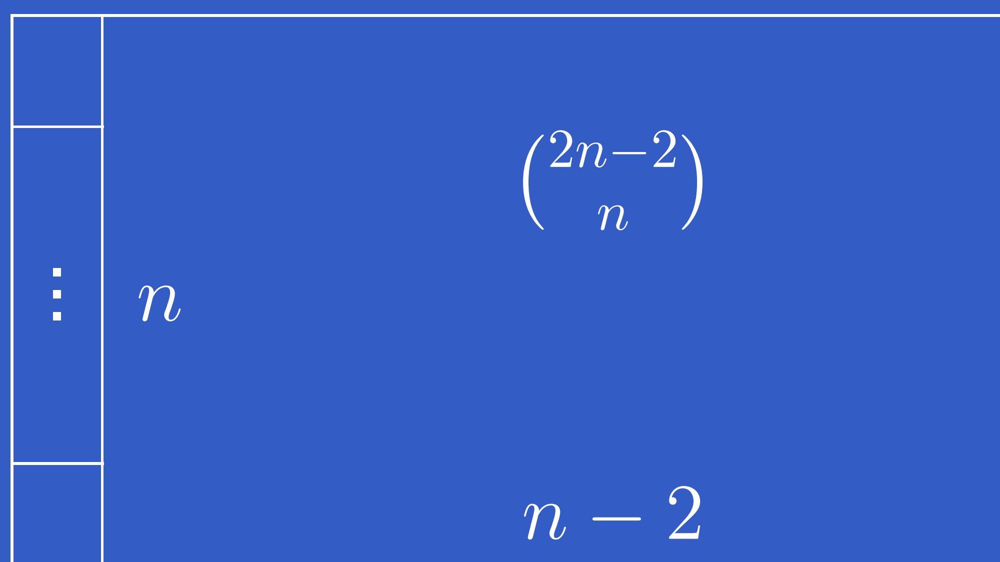
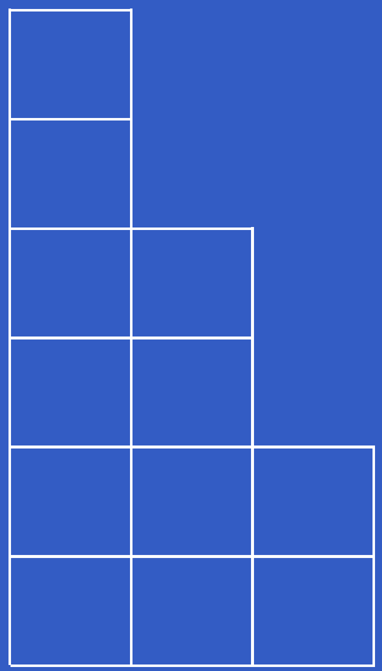
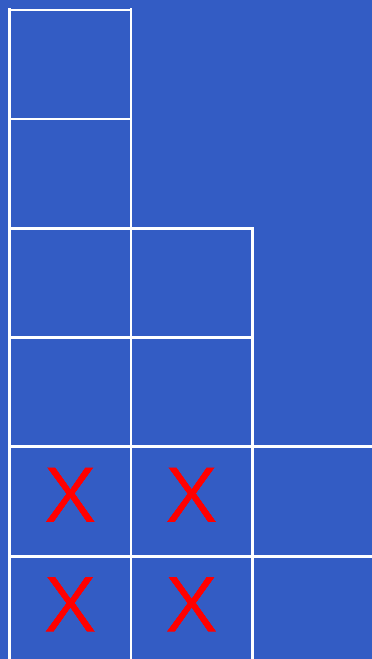


Durfee

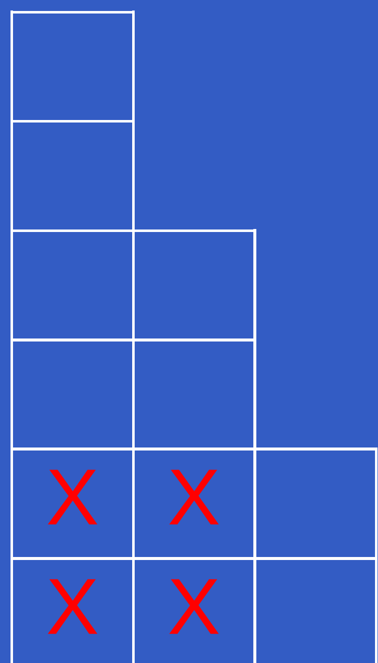


Durfee



Durfee square = $(2, 2)$

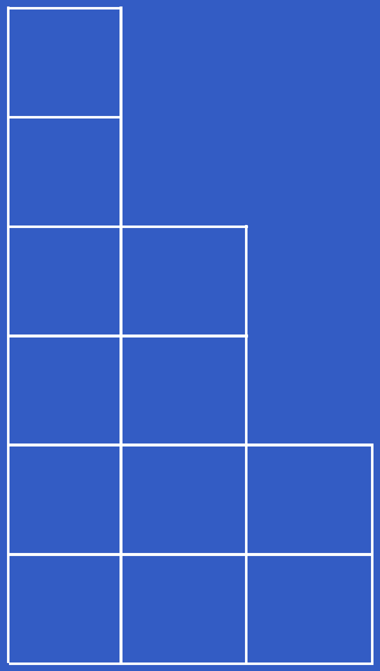
Durfee



Durfee square = $(2, 2)$

Durfee size = 2

Durfee Decomposition



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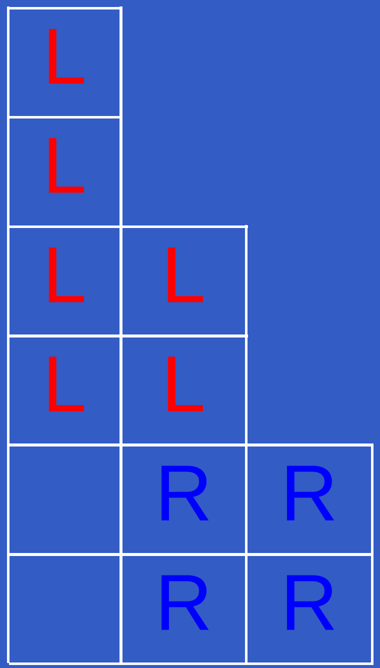

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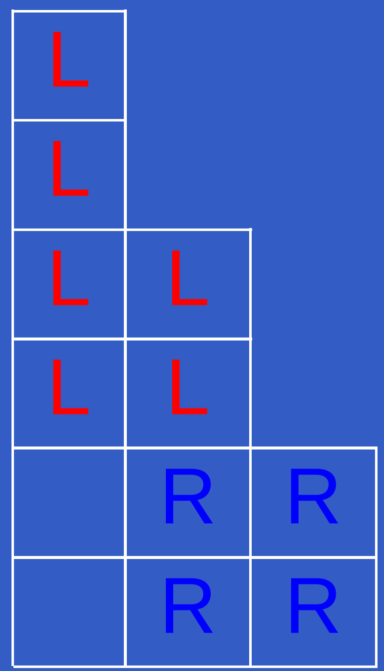

L = $(4, 2)$

R = $(2, 2)$

Dominance Order

The 'natural' partial order on partitions.
Let μ, ν be two partitions

$$\mu \trianglerighteq \nu :\Leftrightarrow \forall k \geq 1 : \sum_{i=1}^{k} \mu_i \geq \sum_{i=1}^{k} \nu_i$$

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Dominance order is compatible with graphical partitions:

$$\nu \text{ graphical}, \mu \trianglerighteq \nu \Rightarrow \mu \text{ graphical}$$

Criterion

Theorem:
A partition λ of even weight is graphical

$$\iff$$

$$L(\lambda) \trianglerighteq R(\lambda)$$

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Recursion Formula (1)

$G(n)$:= set of graphical partitions of length n

$G_s(n)$:= set of graphical partitions of length n and maximal part of size s

$$G(n) = G_1(n) \dot{\cup} \ldots \dot{\cup} G_{n-1}(n)$$

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$$G(n) = G_1(n) \dot\cup \ldots \dot\cup G_{n-1}(n)$$

Each $G_s(n)$ is decomposed into disjoint subsets according to the weight

$$G_s(n) = G_{s,2}(n) \dot\cup \ldots \dot\cup G_{s,n*(n-1)}(n)$$

Recursion Formula (2)

Each set $G_{s,w}(n)$ is decomposed according to the size of the Durfee square

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$$G_{s,w}(n) = G_{s,w,1}(n)\dot{\cup}\ldots\dot{\cup}G_{s,w,n-1}(n)$$

From the Durfee decomposition and the criterion we get a bijection:

$$G_{s,w,d}(n) \longleftrightarrow$$

$$\left\{\ (\mu,\nu) \quad with \quad \begin{array}{c} \mu \trianglerighteq \nu \\ 1 \leq l(\mu) \leq d, \mu_1 = n-d, l(\nu) = id \\ |\nu| + |\mu| = n - (d-1) * d \end{array} \ \right\}.$$

Recursion Formula (3)

$$P(s_1, l_1, w_1, l_2, w_2) \quad := \quad \text{pairs } (\mu, \nu) \text{ with}$$

$$\mu \trianglerighteq \nu,\ \mu_1 = s_1$$

$$l(\mu) = l_1, |\mu| = w_1$$

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$$l(\mu) = l_1, |\mu| = w_1$$
$$l(\nu) = l_2, |\nu| = w_2$$

rewrite above recursion with $r = n - (d-1) * d$:

$$G_{s,w,d}(n) \longleftrightarrow \dot{\bigcup}_{\substack{j=1,\ldots,d \\ l=0,\ldots,r}} P(n-d, j, l, d, r-l)$$

Recursion Formula (4)

L	L		
L	L		
L	L	L	
L	L	L	
		R	R
		R	R
		R	R

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L	L		
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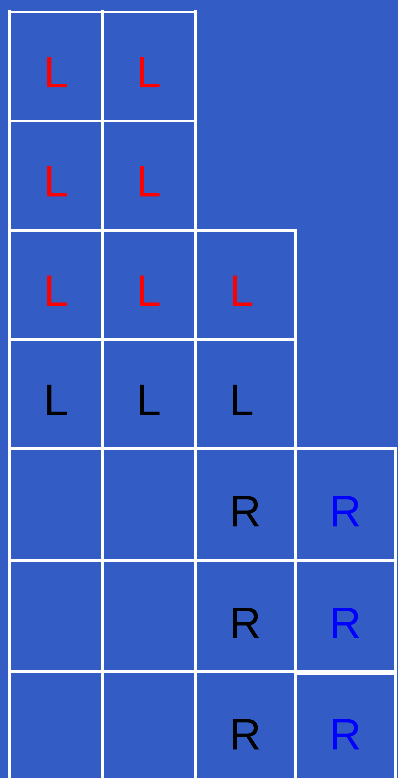
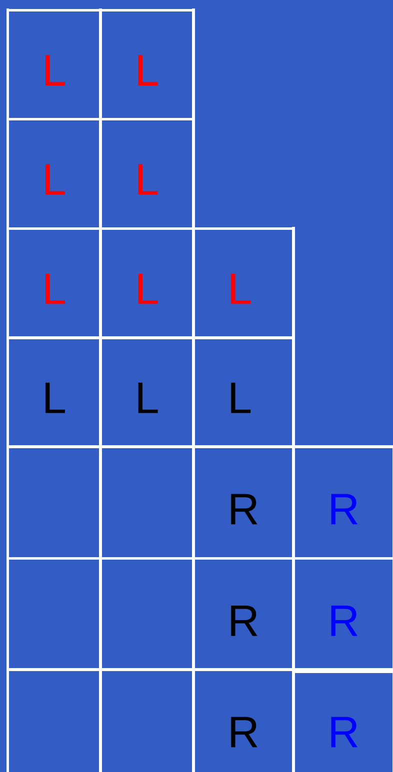


$$P(s_1, l_1, w_1, l_2, w_2)$$

Recursion Formula (4)



$$P(s_1, l_1, w_1, l_2, w_2)$$

$$\updownarrow$$

$$\dot{\bigcup_{\substack{i=0,\ldots,l_1\\ j=0,\ldots,l_2}}} P(s_1 - 1, i, w_1 - l_1, j, w_2 - l_2)$$

Product Formula

We count pairs $\mu \trianglerighteq \nu$, with certain properties

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Unique minimal partition μ^-, unique maximal partition ν^+.
If $\mu^- \trianglerighteq \nu^+$ then (with $p(..) = |P(..)|$)

$$p(s_1, l_1, w_1, l_2, w_2) = p(s_1, l_1, w_1, 0, 0)\left(\sum_{i=1,\ldots,w_2-l_2+1} p(i, l_2, w_2, 0, 0)\right).$$

Results

$g(4), ..$	$..g(19)$	$g(20), \ldots$	
7	162769	7429.160296	
20	614198	28723.877732	
71	2.330537	111236.423288	
240	8.875768	431403.470222	
871	33.924859		
3148	130.038230		
11655	499.753855		
43332	1924.912894		

Results

$g(4), ..$	$..g(19)$	$g(20), ...$	$g(28), ..., g(34)$
7	162769	7429.160296	385.312558.571890
20	614198	28723.877732	1504.105116.253904
71	2.330537	111236.423288	5876.236938.019298
240	8.875768	431403.470222	22974.847399.695092
871	33.924859	1.675316.535350	89891.104720.825873
3148	130.038230	6.513837.679610	351942.828583.179792
11655	499.753855	25.354842.100894	1.378799.828613.947813
43332	1924.912894	98.794053.269694	

Concluding Remarks

Limiting factors:

memory to store intermediate results

time if you do not store intermediate results

References

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- N. Sloane: online database of integer sequences, number A095268
- A. Kohnert: Dominance Order and Graphical Partitions, Electronic Journal of Combinatorics, 2004