A non-free $\mathbb{Z}_4$-linear code of high minimum Lee distance

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Thurnau
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joint work with Michael Kiermaier

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- Not free as $\mathbb{Z}_4$-module.

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- $S := \{0, 1\}$: set of representatives of $\mathbb{Z}_4/\mathrm{Rad}(\mathbb{Z}_4)$.

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- v from above is uniquely determined by $c = v^t\Gamma$ and called *information vector* of c.

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- Due to linearity:

  $$d_{\min}(C) = \min\{w_{\mathrm{Lee}}(c) : 0 \neq c \in C\}.$$

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  is an isometry between $(\mathbb{Z}_4, d_{\mathsf{Lee}})$ and $(\mathbb{F}_2^2, d_{\mathsf{Ham}})$, the *Gray map*.

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- γ transforms any block code $C \subset \mathbb{Z}_4^n$ into a binary code of same size and weights and double length.
- $\mathbb{Z}_4$-linearity of C usually does not lead to $\mathbb{F}_2$-linearity of $\gamma(C)$.

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- The *canonical* one has as first unit a symbol 1 and is denoted by $\kappa(p)$.

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  p_i and l are neighbors $\Leftrightarrow \langle u, v_i \rangle \in \mathrm{Rad}(\mathbb{Z}_4)$.

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- Each neighbor class consists of 4 points.
- Any line intersects 3 different neighbor classes, each in 2 points.

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Lemma

Let $\mathcal{O}$ be a hyperoval in $\mathrm{PHG}(2, \mathbb{Z}_4)$. Then:

- *Each line meets $\mathcal{O}$ in zero or two points. This happens for 7 and 21 lines, respectively.*
- *From each neighbor class there is exactly one point in $\mathcal{O}$.*

Maybe a picture says more than 8 slides...

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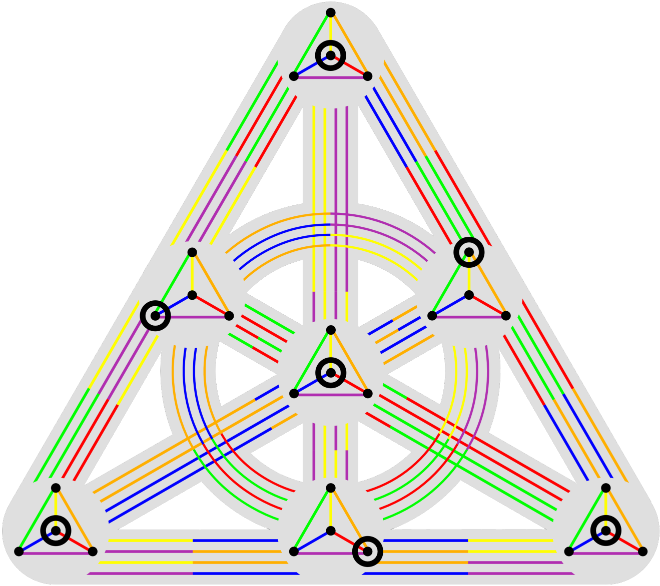

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- Let $\mathcal{O}$ be a hyperoval in $\mathrm{PHG}(2, \mathbb{Z}_4)$ and

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- For a point $p \in \mathcal{P}$ we define a vector

$$v_p = \begin{pmatrix} \kappa(p) \\ \mu(p) \end{pmatrix} \in \mathbb{Z}_4^3 \times \mathrm{Rad}(\mathbb{Z}_4).$$

Construction cont.

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Lemma

Let $\mathcal{P} = \{p_0, \ldots, p_{27}\}$, $\Gamma := (v_{p_0}, \ldots, v_{p_{27}}) \in \mathbb{Z}_4^{(3+1)\times 28}$ *and* C *the code generated by* Γ. *Then:*

$$\mathrm{Lee}_C = 1 + 49X^{26} + 56X^{28} + 7X^{32} + 14X^{34} + X^{42}$$

and the subcode $(\mathbb{Z}_4^3 \times \{0\})\Gamma$ *contains exactly the codewords of Lee weight* 0, 28 *and* 32.

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Let $\delta := (0\ 0\ 0\ 2)^t \in \mathbb{Z}_4^4$ *and* $\hat{\Gamma} := (\Gamma|\delta) \in \mathbb{Z}_4^{(3+1)\times 29}$*. For the code* $\hat{C}$ *generated by* $\hat{\Gamma}$ *holds*

$$\mathrm{Lee}_{\hat{C}} = 1 + 105X^{28} + 7X^{32} + 14X^{36} + X^{44}.$$

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Remark

Claim does not depend on $\mathcal{O}$ *and* $\kappa(-)$. *For example,*

$$\hat{\Gamma} := \begin{pmatrix} 0022\ 0022\ 1111\ 1111\ 0022\ 1111\ 1111\ 0 \\ 0202\ 1111\ 0022\ 1133\ 1111\ 0022\ 1133\ 0 \\ 1111\ 0202\ 0202\ 1313\ 1313\ 1313\ 0202\ 0 \\ 0222\ 0222\ 0222\ 2022\ 2202\ 2202\ 2220\ 2 \end{pmatrix}$$

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- If $2u = 0$: $u = 0$ yields the zero codeword. Otherwise, choose u' with $u = 2u'$. Use $\langle u, p_i \rangle = 2\langle u', p_i \rangle$ and the table above.

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Any $c \in C$ can uniquely be written as $c = (u^t, s)^t \Gamma$. We distinguish a few cases for u and s:

1. $s = 0$:

- If $2u \neq 0$, consider $l := \mathbb{Z}_4 u^\perp \in \mathrm{PHG}(2, \mathbb{Z}_4)$:

$\langle u, \kappa(p_i) \rangle$	0	2	1 or 3
#	6 times	6 times	16 times

  $\rightsquigarrow$ 56 codewords of Lee weight $6 \cdot 2 + 16 \cdot 1 = 28$.
- If $2u = 0$: $u = 0$ yields the zero codeword. Otherwise, choose u' with $u = 2u'$. Use $\langle u, p_i \rangle = 2\langle u', p_i \rangle$ and the table above.
  $\rightsquigarrow$ 7 codewords of Lee weight $16 \cdot 2 = 32$ and one of weight zero.

Proof cont.

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2. $s = 1$:

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   - If $2u \neq 0$, consider $l := \mathbb{Z}_4 u^\perp \in \mathrm{PHG}(2, \mathbb{Z}_4)$.
     - If $l \cap \mathcal{O} = \emptyset$:

$\langle u, \kappa(p_i) \rangle$	0		2		1 or 3	
$\mu(p_i)$	0	2	0	2	0	2
$\#$	$0\times$	$6\times$	$3\times$	$3\times$	$4\times$	$12\times$

Proof cont.

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$\rightsquigarrow$ 14 codewords of Lee weight $9 \cdot 2 + 16 \cdot 1 = 34$.

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  - If $\#(l \cap \mathcal{O}) = 2$:

$\langle u, \kappa(p_i)\rangle$	0		2		1 or 3	
$\mu(p_i)$	0	2	0	2	0	2
#	$2\times$	$4\times$	$1\times$	$5\times$	$4\times$	$12\times$

Proof cont.

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   - If $2u \neq 0$, consider $l := \mathbb{Z}_4 u^\perp \in \mathrm{PHG}(2, \mathbb{Z}_4)$.
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#	0×	6×	3×	3×	4×	12×

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#	2×	4×	1×	5×	4×	12×

$\rightsquigarrow$ 42 codewords of Lee weight $5 \cdot 2 + 16 \cdot 1 = 26$.

Proof cont.

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Continuation for $s = 1$:

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Continuation for $s = 1$:

- If $2u = 0$: $u = 0$ yields the last row of Γ. Otherwise:

$\langle u, \kappa(p_i) \rangle$	0		2	
$\mu(p_i)$	0	2	0	2
$\#$	$3\times$	$9\times$	$4\times$	$12\times$

Proof cont.

Continuation for $s = 1$:

- If $2u = 0$: $u = 0$ yields the last row of Γ. Otherwise:

$\langle u, \kappa(p_i)\rangle$	0		2	
$\mu(p_i)$	0	2	0	2
#	$3\times$	$9\times$	$4\times$	$12\times$

$\rightsquigarrow$ 7 codewords of Lee weight $13 \cdot 2 = 26$ and one of weight 42.

Proof cont.

Continuation for $s = 1$:

- If $2u = 0$: $u = 0$ yields the last row of Γ. Otherwise:

$\langle u, \kappa(p_i)\rangle$	0		2	
$\mu(p_i)$	0	2	0	2
$\#$	$3\times$	$9\times$	$4\times$	$12\times$

$\rightsquigarrow$ 7 codewords of Lee weight $13 \cdot 2 = 26$ and one of weight 42.

$\square$

Questions to the audience

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- Are there more examples of **non-free** $\mathbb{Z}_4$-linear codes where the minimum distance of the Gray image exceeds the linear upper bound?

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Thanks for your attention!